EIGENVALUE ASYMPTOTICS OF NARROW DOMAINS

by

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DEDICATION

To my family, for all their support and love.

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ABSTRACT

We considered the spectrum of the Dirichlet Laplacian Δ_ϵ on the planar domain

$$\Omega_\epsilon = \{(x, y) : -l_1 < x < l_2, 0 < y < \epsilon h(x)]\}$$

where $l_1, l_2 > 0$ and $h(x)$ is a positive analytic function having 0 the only point where it achieves its global maximum M. In particular we studied in details about the full asymptotics of the eigenvalues. First we decompose Δ_ϵ corresponding to the decomposition of the vertical L^2 space into the fundamental mode and remaining higher modes. Then we analyze model operator corresponding to the fundamental mode. In the end we investigate the difference between the model operator and the original Δ_ϵ.

CHAPTER 1

Introduction

1.1 Introduction

In recent years there are a lot of studys about spectrum of narrow domains. Problems of such arise naturally in many areas of physics and mathematics. A nice survey on this topic is by Daniel Grieser [12]. For interesting applications in related mathematical areas see [4], [5], [7], [8], [13], [15], [19]. There are also a lot of applications in mathematical physics as in [16], [18], [20]. For higher dimensional situations see [2]. For discussions on the case for Neumann boundary conditions see [14].

This dissertation is motivated by the work [10] in 2009 where the authors obtained a two-term asymptotics of the eigenvalues for the Dirichlet Laplacian Δ_ϵ in a family of bounded domains $\Omega_\epsilon = \{(x, z) : -l_1 < x < l_2, 0 < z < \epsilon h(x)\}$ where $l_1, l_2 > 0$ and $h(x)$ is a positive analytic function having 0 the only point where it achieves its global maximum M. With such assumptions on $h(x)$, one easily sees $h(x) = M - c(x)x^m$ for some even integer m and some positive analytic function $c(x)$ with $c(0) = c_0 \neq 0$. And the two term asymptotics that was found in [10] is the following:

$$\lim_{\epsilon\to 0}\epsilon^{2\alpha_1}\left(\Lambda_j(\epsilon)-\frac{\pi^2}{M^2\epsilon^2}\right)=\mu_j \tag{1.1}$$

where μ_j are eigenvalues of the operator on $L^2(\mathbf{R})$ given by

$$\mathbf{H}=-\frac{d^2}{dx^2}+q(x), \quad q(x)=\frac{2c_0\pi^2}{M^3}x^m$$

and $\Lambda_j(\epsilon)$ are eigenvalues of the Dirichlet Laplacian Δ_ϵ with $\alpha_1=\frac{2}{m+2}$.

In this dissertation we will focus on finding the formula for the full asymptotics of the eigenvalues $\Lambda_j(\epsilon)$ as $\epsilon\to 0$. Notice in the work [1] Borisov and Freitas considered a very similar problem. In their work by making ansats of boundary layer type localizing in a vicinity of the point 0 they obtained the full asymptotic expansion for the eigenvalues and eigenfunctions. However the method that we will develop in this paper is more operator theoretic, which is very different from what they use in [1]. In particular we are going to decompose the Laplace operator Δ_ϵ as a 2×2 matrix of operators corresponding to the decomposition of the vertical L^2 space into the fundamental mode and the remaining higher modes. Then we analyze the part corresponding to the fundamental vertical mode, which will be called the model operator A_{11}. On top of that we investigate the difference between the eigenvalues of Δ_ϵ and those of the model operator A_{11}.

1.2 Main Results

To fix the notation, let $\Omega_\epsilon = \{(x,y) : -l_1 < x < l_2, 0 < z < \epsilon h(x)]\}$ where $l_1, l_2 > 0$ and $h(x) = M - c(x)x^m$ is a positive analytic function having 0 the only point achieving the global maximum. Here m is some even integer. And we have the taylor expansion $c(x) = c_0 + \sum_{n=1}^{\infty} c_n x^n$.

Let Δ_ϵ be the Dirichlet Laplacian on Ω_ϵ and we are interested in the asymptotic behavior of the spectrum of Δ_ϵ. More precisely, we would consider the following Dirichlet eigenvalue problem

$$\Delta_\epsilon u = \Lambda_\epsilon u \tag{1.2}$$

It is well known that the eigenvalues Λ_ϵ are discrete and tend to infinity. The question we want to address in this paper is:

What is the full asymptotics of the eigenvalues Λ_ϵ as $\epsilon \to 0$?

There are three stages in the whole analysis. The starting point of the whole analysis is to restrict the Dirichlet Laplacian Δ_ϵ to a proper closed subspace and turn the whole problem into a perturbation problem. In particular we have the following Decomopsition Theorem.

Theorem A. *The Dirichlet eigenvalue problem*

$$\Delta_\epsilon u = \Lambda_\epsilon u$$

is equivalent to

$$A_{11}u_1 + A_{12}u_2 = \Lambda_\epsilon u_1$$

$$A_{21}u_1 + A_{22}u_2 = \Lambda_\epsilon u_2$$

where $u_1 = Pu, u_2 = Qu, A_{11} = P\Delta_\epsilon P, A_{12} = P\Delta_\epsilon Q, A_{21} = Q\Delta_\epsilon P, A_{22} = Q\Delta_\epsilon Q,$ *with* P *the othorgonal projection onto*

$$\mathcal{L}_\epsilon = \{u(x,z) = \chi(x)\sqrt{\frac{2}{\epsilon h(x)}}\sin\left(\frac{\pi z}{\epsilon h(x)}\right) : \chi(x) \in H_0^1([-l_1, l_2])\}$$

and $Q = \mathbf{I} - P$.

Following **Theorem A** we study in details about the operator A_{11}, which is essentially a semi-classical one dimensional Schrödinger operator. The key idea here is to introduce scaling $x = \epsilon^{\alpha_1}y, \alpha_1 = \frac{2}{m+2}$. By scalling one will have

$$\epsilon^{2\alpha_1}\left(A_{11} - \frac{\pi^2}{M^2\epsilon^2}\right) = H_0 + \sum_{n=1}^{\infty} H_n\epsilon^{n\alpha_1}$$

where $H_0 = -\frac{d^2}{dz^2} + \frac{2\pi^2 c_0}{M^3}y^m$ is an anharmonic oscillator and H_n is some polynomial in y of degree $n+m$.

Using resolvent expansion one will have the full asymptotic expansion of the eigenvalues for the operator $H_0 + \sum_{n=1}^{\infty} H_n\epsilon^{n\alpha_1}$ thanks to the exponential decaying of eigenfunctions of H_0 and the fact that $H_\epsilon = H_0 + \sum_{n=1}^{\infty} H_n\epsilon^{n\alpha_1}$ defined over

$\mathbf{H}^1([-\frac{l_1}{\epsilon^{\alpha_1}}, \frac{l_2}{\epsilon^{\alpha_1}}])$ can be approximated in a certain sense by $H_{\epsilon,K} = H_0 + \sum_{n=1}^{K} H_n \epsilon^{n\alpha_1}$ defined over $\mathbf{H}^1(\mathbf{R})$. In summary the asymptototics of the eigenvalues λ of the model oeprator A_{11} is stated in the following Theorem.

Theorem B. *Let* $\{\mu_j\}_{j=0}^{\infty}$ *be the full set of eigenvalues of* H_0 *defined on* $\mathbf{H}^1(\mathbf{R})$*. Then* $\epsilon^{2\alpha_1}\left(A_{11} - \frac{\pi^2}{M^2\epsilon^2}\right)$ *has full eigenvalue asymptotics given by*

$$\lambda \sim \mu_j + \sum_{n=1}^{\infty} q_n \epsilon^{n\alpha_1}$$

where q_n *can be computed explicitly.*

The last stage of the work is to understand the difference $\tilde{\lambda} = \Lambda - \lambda$ between the eigenvalues of the original operator and the model operator. It turns out this difference $\tilde{\lambda}$ is the fixed point of an analytic function, which is a contraction on a suitable real interval. Thus one can obtain the difference $\tilde{\lambda}$ using successive approximations. The result is stated as follows.

Theorem C. *Let* λ *be eigenvalues of* A_{11} *with normalized eigenfunction* ϕ*. We also let* $\tilde{\lambda} = \Lambda_\epsilon - \lambda$ *. Then* $\tilde{\lambda}_n \to \tilde{\lambda}$ *as* $n \to \infty$*, where*

$$\tilde{\lambda}_0 = a_0$$

$$\tilde{\lambda}_{n+1} = g(\tilde{\lambda}_n)$$

$a_0 = -\frac{\langle u_1, A_{12}(A_{22}-\lambda)^{-1}A_{21}\phi\rangle}{\langle u_1,\phi\rangle}$, $a_n = -\frac{\langle u_1, A_{12}(A_{22}-\lambda)^{-n-1}A_{21}\phi\rangle}{\langle u_1,\phi\rangle}$ *and* $g(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n$.

As a corollary one obtained an upper bound for the difference $\tilde{\lambda}$.

Theorem D. $|\tilde{\lambda}| = O(\epsilon^{2-2\alpha_1})$. *In particular* $\lim_{\epsilon \to 0} \tilde{\lambda} = 0$.

In this way we have a detailed analysis for the full asymptotics of the Dirichlet Eigenvalues.

1.3 Organization of the Dissertation

This dissertation has three parts. The first part consists of Chapter 2 and reviews some general facts of oeprator theory. In particular we sketch the basic results about Dirichlet Laplacian and Schrödinger operator, which will be used freely in later discussions.

The second part consists of Chapter 3, 4 and 5 and it is the main body of this dissertation work. In particular it is devoted to the proof of **Theorem A** ,**Theroem B** and **Theorem C**. We start Chapter 3 with proving **Theorem A** which serves as the starting point for the whole analysis. In Chapter 4 we investigate model operator in great details. In Section 4.1 we derive an explicit formula for the model operator A_{11} which allows us to study this model operator semi-classically. To study the eigenvalues for the model operator we looked at a special case in Section 4.2 first and then in Section 4.3 we finished with the dicussion for the general case along with the proof for **Theorem B**. In Chapter 5 we prove **Theorem C** and that consists of two parts. In Section 5.1 we derive an equation for the difference $\tilde{\lambda}$. And then in Section 5.2 we provide the successive approximation scheme for solving the equation by proving the map involved is indeed a contraction map. As an easy corollary we also prove **Theorem D**.

The last part consists of the last Chapter and is more speculative. It summarizes the questions left open in this dissertation and consists of further directions for research based on this dissertation .

CHAPTER 2

Preliminaries

In this chapter we review the basic spectral theory of operators along which we nail down the definitions of the related concepts involved in this dissertation. And the notations should be understood within this chapter only and should not be viewed as conflicted with the same notations (if any) in the other Chapters.

2.1 Dirichlet Laplacian

2.1.1 Friedrichs Extension

Definition 2.1.1. A symmetric operator H defined on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ is semibounded if

$$\langle u, Lu \rangle \geq c\|u\|^2$$

for some constant c and all u in $\mathcal{D}$. We also define the space $\mathcal{D}_H$ as the completion of $\mathcal{D}$ with respect to the $H-$norm $\langle u, u \rangle_H = \langle u, Hu \rangle$

Lemma 2.1.2. *The natural map $\mathcal{D}_H \to \mathcal{H}$ is an embedding.*

Proof. See [6]. □

Now we define the Friedrichs extension H_F of H as follows:

For all $v \in \mathcal{D}_H$, $l(v) = \langle v, g \rangle, \forall g \in H$ defines a bounded linear functional on $\mathcal{D}_H$. By Riesz Frechet representation theorem, $l(v) = \langle v, w \rangle_H$ for some $w \in \mathcal{D}_H$. All such w's will be the domain $\mathcal{D}_{H_F}$ of H_F. The operator H_F is defined by

$$H_F w = g, w \in \mathcal{D}_{H_F}$$

In conclusion we have $\langle v, w \rangle_H = \langle v, H_F w \rangle$.

Theorem 2.1.3. *Let H be a semibounded symmetric operator and let H_F be its Friedrich extension. Then H_F is a self-adjoint extension of H.*

Proof. See [6]. □

2.1.2 Dirichlet Laplacian

Let Ω be a region in $\mathbf{R}^N$. We consider the operator on $L^2(\Omega)$ given by $-\Delta$ (negative Laplacian), initially defined over $C_0^\infty(\bar{\Omega})$. By Poincaré Inequlaity, $-\Delta$ thus defined is semibounded. The Friedrichs Extension of $-\Delta$ is the so called Dirichlet Laplacian. The domain of Dirichlet Laplacian is the usual Sobolev space $W_0^{1,2}(\Omega)$, which is the closure of $C_c^\infty(\Omega)$ with respect to inner product

$$\langle f, g \rangle_1 = \int_\Omega \Big(f(x)\bar{g}(x) + \nabla f(x) \cdot \overline{\nabla g(x)} \Big)\, dx.$$

The spectral property we need about Dirichlet Laplacian is the following:

Theorem 2.1.4. *Let Ω be a bounded region in $\mathbf{R}^N$. Let $H = -\Delta$ be the Dirichlet Laplacian acting on $L^2(\Omega)$. Then H has empty essential spectrum and compact resolvent. In particular the eigenvalues of H are*

$$0 < \lambda_1 \leq \lambda_2 \cdots < \infty$$

with the corresponding eigenfunctions $\{f_n\}$ forming a complete orthonormal set.

Proof. See [6]. □

Example 2.1.5. Let Ω be the cube

$$\{x = (x_1, x_2, \cdots, x_N) : 0 < x_i < a, i = 1, 2, \cdots N\}$$

Then the functions

$$f_n(x) = \left(\frac{2}{a}\right)^{N/2} \prod_{i=1}^{N} \sin(\pi n_i x_i / a)$$

parametrized by the multi-index of positive integers $n = (n_1, n_2, \cdots, n_N)$ form a complete orthonormal set of eigenfunctions of Dirichlet Laplacian with corresponding eigenvalues

$$\lambda_n = \pi^2 a^{-2}(n_1^2 + n_2^2 + \cdots + n_N^2)$$

The operator $H = -\Delta$ has compact resolvent.

2.2 Schrödinger Operator

2.2.1 Schrödinger Operator on the Real Line

Consider an operator H_0 defined on $C_0^\infty(\mathbf{R})$ by the formula

$$H_0 u = -u'' + V(x)u$$

where $V(x) \in L^\infty_{loc}$ is a real valued function. Clearly H_0 is a symmetric operator in $L^2(\mathbf{R})$. Recall that H_0 is said to be essentially self-adjoint if its closure H_0^{**} is a self adjoint operator. In this case H_0 has one and only one self-adjoint extension and its spectral property is simple as shown in the next theorem.

Theorem 2.2.1. *Assume $V(x) \in L^\infty_{loc}(\mathbf{R})$ is a real valued function and*

$$\lim_{x\to\infty} V(x) = +\infty.$$

Then H_0 is essentially self-adjoint. Denote by H the closure of H_0. The spectrum of H is discrete. In particular there exists an orthonormal system $\phi_i(x), i = 0, 1, \cdots$, of eigenfunctions with eigenvalues $\lambda_i \to +\infty$ as $i \to \infty$. Moreover all the eigenvalues are simple. If $\lambda_0 < \lambda_1 < \cdots$, then any (nontrivial) eigenfunction corresponding to λ_k has exactly k nodes. All eigenfunctions decay exponentially fast at infinity.

Proof. See [3]. □

Remark 2.2.2. The spectrum of H in the above theorem is discrete if and only if

$$\int_r^{r+1} V(x)dx \to +\infty \quad as \quad r \to \infty.$$

Theorem 2.2.3. *Assume $V(x) \in L^\infty_{loc}(\mathbf{R}^n)$ and $V(x) \geq -C$. We also assume*

$$\liminf_{x\to\infty} V(x) \geq a$$

Let ψ be an eigenfunction of H with the eigenvalue $\lambda < a$. Then for every $\epsilon > 0$ there exits $C_\epsilon > 0$ such that

$$|\psi(x)| \leq C_\epsilon \exp\left(-\sqrt{\frac{a-\lambda-\epsilon}{2}}|x|\right)$$

Corollary 2.2.4. *If $V(x) \to +\infty$ as $|x| \to +\infty$ then*

$$|\psi(x)| \leq C_a e^{-a|x|}$$

for every $a > 0$.

2.2.2 Schrödinger Operator on Bounded Interval

Schrödinger Operator on Bounded Interval can be treated easily under the well developed Sturm-Liouville thoery. And the main result we need is the following.

Theorem 2.2.5. *Assume $V(x) \in L^1([a,b])$, then $-\frac{d^2}{dx^2}+V(x)$ on $[a,b]$ with Dirichlet Boundary condition has discrete spectrum consisting only simple eigenvalues*

$$-\infty < \lambda_0 < \lambda_1 < \cdots .$$

Proof. See [17]. $\square$

2.3 Asymptotics

In this section, we will give the basic definition of **Asymptotic Expansion** and provide some interesting examples.

Definition 2.3.1 (Asymptotic Scale)**.** Let $\{f_n\}$ be a sequence of continuous functions on some domain and L is a limit point of the domain. Then the sequence consititues an asymptotic scale if for every n, $f_{n+1}(x) = o(f_n(x))$ as $x \to L$.

Definition 2.3.2 (Asymptotic Expansion)**.** Let f be a function on the domain of the asymptotic scale $\{f_n\}$, then f has an **asymptotic expansion of order** N with respect to the scale as a formal series $\sum_{n=0}^{N} a_n f_n(x)$ if

$$f(x) - \sum_{n=0}^{N-1} a_n f_n(x) = O(f_N(x)), \quad x \to L.$$

If this holds for all N, then we say f has a **full asymptotic expansion** and we write

$$f(x) \sim \sum_{n=0}^{\infty} a_n f_n(x), \quad x \to L.$$

Example 2.3.3. (See [23]) Suppose $\phi(t)$ is of rapidly decay at infinity and is smooth around 0. Then $\phi(t) \sim \sum_{n=0}^{\infty} a_n t^n$ as $t \to 0$. Moreover the Mellin transform $\tilde{\phi}(s)$ of $\phi(t)$

$$\tilde{\phi}(s) = \int_0^{\infty} \phi(t) t^{s-1} dt$$

has a meromorphic continuation to all of $\mathbf{C}$ with simple poles of residue a_n at $s = -n(n = 0, 1, 2, \cdots)$ and no other poles.

Example 2.3.4. (See [23]) Let $f : [0, \infty) \to \mathbf{C}$ be a smooth function together with all its derivatives are of sufficiently rapid decay at infinity such that $g(x) = f(x) + f(2x) + f(3x) + \cdots$ converges and

$$f(x) \sim \sum_{n=0}^{\infty} b_n x^n \quad (x \to 0)$$

Then as $x \to 0$, we have

$$g(x) \sim \frac{\int_0^\infty f(t)dt}{x} + \sum_{n=0}^{\infty} (-1)^n b_n \frac{B_{n+1}}{n+1} x^n$$

CHAPTER 3

Decomposition of Dirichlet Laplacian Δ_ϵ

In this chapter we start the analysis of the problem stated in the Introduction. Recall

$$\Omega_\epsilon = \{(x,y) : -l_1 < x < l_2, 0 < z < \epsilon h(x)]\}$$

where $l_1, l_2 > 0$ and $h(x) = M - c(x)x^m$ is a positive analytic function having 0 the only point achieving the global maximum. Here m is some even integer and we have the taylor expansion $c(x) = c_0 + \sum_{n=1}^{\infty} c_n x^n$.

Let Δ_ϵ be the Dirichlet Laplacian on Ω_ϵ as introduced in **Section 2.1**. We are interested in the Dirichlet Eigenvalue Problem

$$\Delta_\epsilon u = \Lambda_\epsilon u$$

From **Theorem 2.1.4** we know the eigenvalues Λ_ϵ are discrete real numbers and tend to $+\infty$ with the corresponding eigenfunctions forming a basis for $L^2(\Omega_\epsilon)$. The question we want to solve here is

What is the full asymptotics of the eigenvalues Λ_ϵ as $\epsilon \to 0$?

Due to the adiabatic nature of the problem let's consider the following subspace of $H_0^1(\Omega_\epsilon)$,

$$\mathcal{L}_\epsilon = \{u(x,y) = \chi(x)\sqrt{\frac{2}{\epsilon h(x)}}\sin(\frac{\pi y}{\epsilon h(x)}) : \chi(x) \in H_0^1([-l_1, l_2])\}$$

where $H_0^1(\Omega_\epsilon), H_0^1([-l_1, l_2])$ are the usual Sobolev Spaces and $H_0^1(\Omega_\epsilon)$ is also the natural domain of our Dirichlet Laplacian .

It is clear that $\mathcal{L}_\epsilon$ is a closed linear subspace of $H_0^1(\Omega_\epsilon)$. Let P be the orthogonal projection onto $\mathcal{L}_\epsilon$. We also let Q be the orthogonal projection onto the complement of $\mathcal{L}_\epsilon$. Then clearly $P + Q = I$.

With these projections P and Q, we have a decomposition of our Dirichlet Laplacian as follows:

$$\Delta_\epsilon = A_{11} + A_{12} + A_{21} + A_{22}$$

where $A_{11} = P\Delta_\epsilon P, A_{12} = P\Delta_\epsilon Q, A_{21} = Q\Delta_\epsilon P$ and $A_{22} = Q\Delta_\epsilon Q$. This decomposition allows us to rephrase our original eigenvalue problem as an equivalent one as shown in the following lemma.

Lemma 3.0.1. *The Dirichlet eigenvalue problem*

$$\Delta_\epsilon u = \Lambda_\epsilon u$$

is equivalent to

$$A_{11}u_1 + A_{12}u_2 = \Lambda_\epsilon u_1$$

$$A_{21}u_1 + A_{22}u_2 = \Lambda_\epsilon u_2$$

where $u_1 = Pu, u_2 = Qu, A_{11} = P\Delta_\epsilon P, A_{12} = P\Delta_\epsilon Q, A_{21} = Q\Delta_\epsilon P, A_{22} = Q\Delta_\epsilon Q,$ *with* P *the othorgonal projection onto*

$$\mathcal{L}_\epsilon = \{u(x,y) = \chi(x)\sqrt{\frac{2}{\epsilon h(x)}}\sin(\frac{\pi y}{\epsilon h(x)}) : \chi(x) \in H_0^1([-l_1, l_2])\}$$

and $Q = \mathbf{I} - P$.

Proof. Follows directly from definition. $\square$

CHAPTER 4

Study of the Model Operator A_{11}

4.1 Model Operator A_{11} and Its Refinement A_ϵ

Recall that $A_{11} = P\Delta_\epsilon P$ and P is the orthogonal projection onto the closed subspace $\mathcal{L}_\epsilon$. We now derive an explicit formual for A_{11} which will allow us to make further analysis of the spectrum of it.

Theorem 4.1.1 (Explicit Formula of A_{11})**.** *A_{11} is defined on $H_0^1(\mathbf{I})$ with Dirichlet Boundary condition as*

$$A_{11} = -\frac{d^2}{dx^2} + \frac{\pi^2}{\epsilon^2 h^2} + (\frac{\pi^2}{3} + \frac{1}{4})\frac{h'^2}{h^2}$$

where $\mathbf{I} = [-l_1, l_2]$.

Proof. Notice the energy form associated with A_{11} is

$$\begin{aligned}\mathcal{E}(u) &= \langle A_{11}u, u\rangle = \langle P\Delta_\epsilon Pu, u\rangle = \langle \Delta_\epsilon Pu, Pu\rangle \\ &= \int_{\Omega_\epsilon} |\nabla Pu|^2 dxdy\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ represents the L^2 inner product. Now by the definition of P one may assume $Pu = \chi(x)\sqrt{\frac{2}{\epsilon h(x)}}\sin(\frac{\pi z}{\epsilon h(x)})$ where $\chi(x) \in H_0^1(\mathbf{I})$. Then by direct calculation

$$\mathcal{E}(u) = \int_{\mathbf{I}} |\chi'|^2 dx + \int_{\mathbf{I}} [\frac{\pi^2}{\epsilon^2 h^2} + (\frac{\pi^2}{3} + \frac{1}{4})\frac{h'^2}{h^2}]\chi^2 dx$$

where $\mathbf{I} = [-l_1, l_2]$. Clearly the associated differential operator is

$$A_{11} = -\frac{d^2}{dx^2} + \frac{\pi^2}{\epsilon^2 h^2} + (\frac{\pi^2}{3} + \frac{1}{4})\frac{h'^2}{h^2}$$

defined on $H_0^1(\mathbf{I})$ with Dirichlet Boundary condition.

□

Corollary 4.1.2.

$$A_{11} \geq \frac{\pi^2}{\epsilon^2 M^2}$$

Proof. From the equality in the previous theorem, namely

$$\mathcal{E}(u) = \int_{\mathbf{I}} |\chi'|^2 dx + \int_{\mathbf{I}} [\frac{\pi^2}{\epsilon^2 h^2} + (\frac{\pi^2}{3} + \frac{1}{4})\frac{h'^2}{h^2}]\chi^2 dx$$

and the fact that $0 < h(x) \leq M$, the corollary follows. □

It is known [17] that spectrum of A_{11} consists only of eigenvalues. Clearly they depend on ϵ. Now we will look at the dependence on ϵ. More precisely we want to find the full aymptotics of the eigenvalues as $\epsilon \to 0$. By Corollary 4.1.2 it is convenient to subtract the bottom of the spectrum from the engery form to get the following associated differential operator

$$A_\epsilon = -\frac{d^2}{dx^2} + \frac{\pi^2}{\epsilon^2 h^2} - \frac{\pi^2}{\epsilon^2 M^2} + (\frac{\pi^2}{3} + \frac{1}{4})\frac{h'^2}{h^2} \tag{4.1}$$

defined on $H_0^1(\mathbf{I})$ with Dirichlet Boundary condition. The relation between A_ϵ and A_{11} is as follows:

Lemma 4.1.3. $A_\epsilon = A_{11} - \frac{\pi^2}{\epsilon^2 M^2}$.

Proof. Follows from definition. □

Remark 4.1.4. Both A_ϵ and A_{11} are positive operators. In particular the spectrum of both operators consists only eigenvalues [17] .

Now the study of eigenvalue asymptotics for model operator A_{11} becomes the study of the same problem for our refined operator A_ϵ. To show the essential ingredients for the analysis we look at a particular case first and then we will handle the genral case. To fix the notation we consider the following Dirichlet eigenvalue problem

$$A_\epsilon \phi(x) = \nu \phi(x)$$

where $\phi(x) \in H_0^1(\mathbf{I})$.

By **Remark 4.1.4** we may assume the complete set of eigenvlaues for A_ϵ as $\{\nu_j\}_{j=0}^\infty$ with the corresponding L^2 normalized eigenfunctions $\{\phi_j(x)\}_{j=0}^\infty$. For the sake of concision we denote the eigenvalue as ν with the sub-index abbreviated. Similarly for the corresponding eigenfunction.

4.2 Eigenvalue Asymptotics of A_ϵ in a particular case

The particular case we look at is $h(x) = M - c_0 x^m, x \in \mathbf{I} = [-l_1, l_2]$, m an even integer. $h(x)$ is also assumed to be positve on $[-l_1, l_2]$. One thing to notice is that the condition $h(x)$ achieves its global maximum only at 0 is naturally satisfied.

To make progress in the study of the eigenvalues of A_ϵ, the observation to make is A_ϵ is essentially a semi-classical one dimensional Schrödinger operator. Using techniques from semi-classical analysis one might make progress. In particualr, to find the eigenvalue asymptotics we are going to make connections with the even more familiar object, namely (an)harmonic oscillators, by Taylor expansion. Also to make the perturbation regular we introduce a proper scaling.

Lemma 4.2.1. *Let $x = \epsilon^{\alpha_1} y$ where $\alpha_1 = \frac{2}{m+2}, y \in \mathbf{I}_\epsilon = [-\frac{l_1}{\epsilon^{\alpha_1}}, \frac{l_2}{\epsilon^{\alpha_1}}]$, then*

$$\epsilon^{2\alpha_1} A_\epsilon \text{ is unitarily equivalent to } H_\epsilon = H_0 + \sum_{n=1}^{\infty} H_n \epsilon^{n\alpha}$$

with $\alpha = m\alpha_1 = \frac{2m}{m+2}, a_0 = \frac{\pi^2}{M^2}, a_1 = \frac{c_0}{M}, a = (\frac{\pi^2}{3} + \frac{1}{4})\frac{m^2 c_0^2}{\pi^2}$ and $H_0 = -\frac{d^2}{dy^2} + 2a_0 a_1 y^m, H_n = (n+2)a_0 a_1^{n+1} y^{nm+m} + (n-1) a a_0 a_1^{n-2} y^{nm-2}$ are polynomial in y of degree $nm + m$.

Proof. Notice that $h(x) = M - cx^m$ and $\frac{\pi^2}{\epsilon^2 h^2} = \frac{\pi^2}{M^2 \epsilon^2}\left[\sum_{n=1}^{\infty} n \left(\frac{cx^m}{M}\right)^{n-1}\right]$, so we have

$$\begin{aligned} A_\epsilon &= -\frac{d^2}{dx^2} + \frac{\pi^2}{\epsilon^2 h^2} - \frac{\pi^2}{\epsilon^2 M^2} + (\frac{\pi^2}{3} + \frac{1}{4})\frac{h'^2}{h^2} \\ &= -\frac{d^2}{dx^2} + \frac{\pi^2}{M^2\epsilon^2}\left[\sum_{n=2}^{\infty} n \left(\frac{cx^m}{M}\right)^{n-1}\right] + (\frac{\pi^2}{3} + \frac{1}{4})\frac{m^2 c^2}{M^2} x^{2(m-1)} \left[\sum_{n=1}^{\infty} n \left(\frac{cx^m}{M}\right)^{n-1}\right] \end{aligned}$$

Buy introducing $x = \epsilon^{\alpha_1} y$ where $\alpha_1 = \frac{2}{m+2}, y \in \mathbf{I}_\epsilon = [-\frac{l_1}{\epsilon^{\alpha_1}}, \frac{l_2}{\epsilon^{\alpha_1}}]$, we see that

$$A_\epsilon = \frac{1}{\epsilon^{2\alpha_1}}\left(-\frac{d^2}{dy^2} + \frac{\pi^2}{M^2}\epsilon^{2\alpha_1 - 2}\left[\sum_{n=2}^{\infty} n \left(\frac{c(\epsilon^{\alpha_1 - 1} y)^m}{M}\right)^{n-1}\right] + (\frac{\pi^2}{3} + \frac{1}{4})\frac{m^2 c^2}{M^2}(\epsilon^{\alpha_1} y)^{2m} y^{-2}\left[\sum_{n=1}^{\infty} n \left(\frac{c(\epsilon^{\alpha_1} y)^m}{M}\right)^{n-1}\right]\right)$$

Now let $\alpha = m\alpha_1 = \frac{2m}{m+2}, a_0 = \frac{\pi^2}{M^2}, a_1 = \frac{c_0}{M}, a = (\frac{\pi^2}{3} + \frac{1}{4})\frac{m^2 c_0^2}{\pi^2}$. Then

$$\begin{aligned} A_\epsilon &= \frac{1}{\epsilon^{2\alpha_1}}\left(-\frac{d^2}{dy^2} + \frac{\pi^2}{M^2}\epsilon^{2\alpha_1 - 2}\left[\sum_{n=2}^{\infty} n \left(\frac{c(\epsilon^{\alpha_1} y)^m}{M}\right)^{n-1}\right] + (\frac{\pi^2}{3} + \frac{1}{4})\frac{m^2 c^2}{M^2}(\epsilon^{\alpha_1} y)^{2m} y^{-2}\left[\sum_{n=1}^{\infty} n \left(\frac{c(\epsilon^{\alpha_1} y)^m}{M}\right)^{n-1}\right]\right) \\ &= \frac{1}{\epsilon^{2\alpha_1}}\left(-\frac{d^2}{dy^2} + 2a_0 a_1 y^m + \sum_{n=1}^{\infty}\left[(n+2)a_0 a_1^{n+1} y^{nm+m} + (n-1) a a_0 a_1^{n-2} y^{nm-2}\right]\epsilon^{n\alpha}\right) \end{aligned}$$

The Lemma clearly follows by matching the corresponding terms. $\square$

Remark 4.2.2. The convergence of the infinite sum in the proof of **Lemma 4.2.1** is fine since we are essentially just taylor expanding an analytic function followed by a scaling.

Now with **Lemma 4.2.1** we can turn the study of the eigenvalues of A_ϵ into the study of the eigenvalues of $H_\epsilon = H_0 + \sum_{n=1}^{\infty} H_n \epsilon^{n\alpha}$ defined over $H_0^1(\mathbf{I}_\epsilon)$ with $\mathbf{I}_\epsilon = [-\frac{l_1}{\epsilon^{\alpha_1}}, \frac{l_2}{\epsilon^{\alpha_1}}]$. Notice when $\mathbf{I}_\epsilon \to \mathbf{R}$ as $\epsilon \to 0$. This motivates us to consider the anharmonic oscillator $\tilde{H}_0 = -\frac{d^2}{dy^2} + 2a_0a_1y^m$ defined on $\mathbf{R}$. For further discussion, let $\tilde{H}_n = (n+2)a_0a_1^{n+1}y^{nm+m} + (n-1)aa_0a_1^{n-2}y^{nm-2}$ which is the same as H_n except that $\tilde{H}_n$ is defined on $\mathbf{R}$. We also let $\tilde{H}_{\epsilon,K} = \tilde{H}_0 + \sum_{n=1}^{K} \tilde{H}_n\epsilon^{n\alpha}$. To continue let's recall the standard fact about (an)harmonic oscillator.

Lemma 4.2.3. *$\tilde{H}_0 = -\frac{d^2}{dy^2} + 2a_0a_1y^m, m$ an even integer, defined on $H_0^1(\mathbf{R})$ has only discrete eigenvalues $\mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots < \infty$ with the corresponding eigenfunctions $\{\psi_j\}_{j=0}^{\infty}$ forming a complete basis for $H_0^1(\mathbf{R})$.*

Proof. See [3]. $\square$

One can now obtain information of the eigenvalues for $\tilde{H}_{\epsilon,K} = \tilde{H}_0 + \sum_{n=1}^{K} \tilde{H}_n\epsilon^{n\alpha}$ defined on $H_0^1(\mathbf{R})$ using regular perturbation theory. In particular we have

Lemma 4.2.4. *Let $\{\mu_j\}_{j=0}^{\infty}$ be the full set of eigenvalues of $\tilde{H}_0$ defined on $H_0^1(\mathbf{R})$ with corresponding eigenfunctions $\{\psi_j\}_{j=0}^{\infty}$. Let $\lambda(\tilde{H}_{\epsilon,K})$ be the eigenvalue for $\tilde{H}_{\epsilon,K} =$*

$\tilde{H}_0+\sum_{n=1}^{K}\tilde{H}_n\epsilon^{n\alpha}$ *defined on* $H_0^1(\mathbf{R})$ *near* μ_j *with corresponding normalized eigenvector* $\phi_{\epsilon,K}$. *Then*

$$\lambda(\tilde{H}_{\epsilon,K})\sim\mu_j+\sum_{n=1}^{\infty}\epsilon^{n\alpha}\tilde{q}_n \tag{4.2}$$

where

$$\tilde{q}_n=\sum_{k=1}^{n}\frac{1}{2\pi\mathrm{i}}\mathbf{Tr}\int_\Gamma\left(\sum_{j_1+j_2+\cdots+j_k=n,j_i\le K,j_i\in\mathbf{Z}^+}(-1)^k\lambda(\tilde{H}_0-\lambda)^{-1}H_{j_1}(\tilde{H}_0-\lambda)^{-1}\cdots H_{j_k}(\tilde{H}_0-\lambda)^{-1}\right)d\lambda$$

and $\Gamma=\{\lambda:|\lambda-\mu_j|=\delta\}$ *any closed curve enclosing* μ_j *and inside which* $H_{\epsilon,K}$ *has single eigenvalue.*

Proof.

$$\begin{aligned}
&\lambda(\tilde{H}_{\epsilon,K})\\
&=\frac{1}{2\pi\mathrm{i}}\mathbf{Tr}\int_\Gamma\lambda(\tilde{H}_{\epsilon,K}-\lambda)^{-1}d\lambda\\
&=\frac{1}{2\pi\mathrm{i}}\mathbf{Tr}\int_\Gamma\lambda(\tilde{H}_{\epsilon,K}-H_0+H_0-\lambda)^{-1}d\lambda\\
&=\frac{1}{2\pi\mathrm{i}}\mathbf{Tr}\int_\Gamma\lambda(H_0-\lambda)^{-1}\left[\sum_{k=0}^{N}(-1)^k[\left(\tilde{H}_{\epsilon,K}-H_0\right)(H_0-\lambda)^{-1}]^k+(-1)^{(N+1)}\left(I+[(\tilde{H}_{\epsilon,K}-H_0)(H_0-\lambda)^{-1}]\right)^{-1}[(\tilde{H}_{\epsilon,K}-H_0)(H_0-\lambda)^{-1}]^{N+1}\right]d\lambda\\
&=\sum_{k=0}^{N}\frac{1}{2\pi\mathrm{i}}\mathbf{Tr}\int_\Gamma(-1)^k\lambda(H_0-\lambda)^{-1}\left(\sum_{n=k}^{\infty}\epsilon^{n\alpha}\sum_{j_1+j_2+\cdots+j_k=n,j_i\le K,j_i\in\mathbf{Z}^+}H_{j_1}(H_0-\lambda)^{-1}H_{j_2}(H_0-\lambda)^{-1}\cdots H_{j_k}(H_0-\lambda)^{-1}\right)d\lambda\\
&+\sum_{n=N+1}^{\infty}\epsilon^{n\alpha}\frac{1}{2\pi\mathrm{i}}\mathbf{Tr}\int_\Gamma\sum_{j_1+j_2+\cdots+j_{N+1}=n,j_i\le K,j_i\in\mathbf{Z}^+}(-1)^{N+1}\lambda(\tilde{H}_{\epsilon,K}-\lambda)^{-1}H_{j_1}(H_0-\lambda)^{-1}H_{j_2}(H_0-\lambda)^{-1}\cdots H_{j_{N+1}}(H_0-\lambda)^{-1}d\lambda\\
&=\mu_0+\sum_{n=1}^{N}\tilde{q}_n\epsilon^{n\alpha}\\
&+\sum_{n=N+1}^{\infty}\epsilon^{n\alpha}\sum_{k=1}^{N}\frac{1}{2\pi\mathrm{i}}\mathbf{Tr}\int_\Gamma\left(\sum_{j_1+j_2+\cdots+j_k=n,j_i\le K,j_i\in\mathbf{Z}^+}(-1)^k\lambda(H_0-\lambda)^{-1}H_{j_1}(H_0-\lambda)^{-1}H_{j_2}(H_0-\lambda)^{-1}\cdots H_{j_k}(H_0-\lambda)^{-1}\right)d\lambda\\
&+\sum_{n=N+1}^{\infty}\epsilon^{n\alpha}\frac{1}{2\pi\mathrm{i}}\mathbf{Tr}\int_\Gamma\sum_{j_1+j_2+\cdots+j_{N+1}=n,j_i\le K,j_i\in\mathbf{Z}^+}(-1)^{N+1}\lambda(\tilde{H}_{\epsilon,K}-\lambda)^{-1}H_{j_1}(H_0-\lambda)^{-1}H_{j_2}(H_0-\lambda)^{-1}\cdots H_{j_{N+1}}(H_0-\lambda)^{-1}d\lambda
\end{aligned}$$

Hence we have

$$
\begin{aligned}
&\lambda(\tilde{H}_{\epsilon,K}) - \left(\mu_0 + \sum_{n=1}^{N} \tilde{q}_n \epsilon^{n\alpha}\right) \\
&= \sum_{n=N+1}^{\infty} \epsilon^{n\alpha} \sum_{k=1}^{N} \frac{1}{2\pi \mathrm{i}} \mathbf{Tr} \int_\Gamma \left(\sum_{j_1+j_2+\cdots+j_k=n, j_i \le K, j_i \in \mathbf{Z}^+} (-1)^k \lambda (H_0-\lambda)^{-1} H_{j_1} (H_0-\lambda)^{-1} H_{j_2} (H_0-\lambda)^{-1} \cdots H_{j_k} (H_0-\lambda)^{-1} \right) d\lambda \\
&+ \sum_{n=N+1}^{\infty} \epsilon^{n\alpha} \frac{1}{2\pi \mathrm{i}} \mathbf{Tr} \int_\Gamma \sum_{j_1+j_2+\cdots+j_{N+1}=n, j_i \le K, j_i \in \mathbf{Z}^+} (-1)^{N+1} \lambda (\tilde{H}_{\epsilon,K}-\lambda)^{-1} H_{j_1} (H_0-\lambda)^{-1} H_{j_2} (H_0-\lambda)^{-1} \cdots H_{j_{N+1}} (H_0-\lambda)^{-1} d\lambda \\
&= \sum_{n=N+1}^{KN} \epsilon^{n\alpha} \sum_{k=1}^{N} \frac{1}{2\pi \mathrm{i}} \mathbf{Tr} \int_{\Gamma=\{|\lambda-\lambda_0|\le\epsilon\}} \left(\sum_{j_1+j_2+\cdots+j_k=n, j_i \le K, j_i \in \mathbf{Z}^+} (-1)^k \lambda (H_0-\lambda)^{-1} H_{j_1} (H_0-\lambda)^{-1} H_{j_2} (H_0-\lambda)^{-1} \cdots H_{j_k} (H_0-\lambda)^{-1} \right) d\lambda \\
&+ \sum_{n=N+1}^{K(N+1)} \epsilon^{n\alpha} \frac{1}{2\pi \mathrm{i}} \mathbf{Tr} \int_{\Gamma=\{|\lambda-\lambda_0|\le\epsilon\}} \sum_{j_1+j_2+\cdots+j_{N+1}=n, j_i \le K, j_i \in \mathbf{Z}^+} (-1)^{N+1} \lambda (\tilde{H}_{\epsilon,K}-\lambda)^{-1} H_{j_1} (H_0-\lambda)^{-1} H_{j_2} (H_0-\lambda)^{-1} \cdots H_{j_{N+1}} (H_0-\lambda)^{-1} d\lambda \\
&= O(\epsilon^{(N+1)\alpha})
\end{aligned}
$$

The main reason for the last equality is that $\|(\tilde{H}_{\epsilon,K} - \lambda)^{-1} - (\tilde{H}_0 - \lambda)^{-1}\| \to 0$ as $\epsilon \to 0$, which we now prove in the corollary after the following proposition. $\square$

Proposition 4.2.5. *Let* $\mathbf{T} \ge 0$ *and* $\mathbf{T}_\epsilon \ge 0, 0 < \epsilon \le \epsilon_0$ *be bounded selfadjoint operators in a separable Hilbert space and let* $\mathbf{T}_\epsilon \to \mathbf{T}$ *strongly as* $\epsilon \to 0$*. Let* $\mathcal{C}$ *be the ideal class of all compact operators in the algebra of all bounded operators. Suppose also that there exists a bounded selfadjoint operator* $\mathbf{T}_0$ *such that* $\mathbf{T}_\epsilon \le \mathbf{T}_0$ *for all* $\epsilon \le \epsilon_0$ *and*

$$dist(\mathbf{T}_0, \mathcal{C}) = d, d \ge 0.$$

Then

$$\limsup_{\epsilon \to 0} \|\mathbf{T}_\epsilon - \mathbf{T}\| \le d$$

Proof. See [11] Page 112. □

Corollary 4.2.6.

1. *Let $V(x)$ and $V_\epsilon(x), 0 < \epsilon \leq \epsilon_0$ be nonnegative measurable functions on $\mathbf{R}$, such that $V(x) \to \infty$ as $|x| \to \infty$ and*

$$V_\epsilon(x) \to V(x) \text{ as } \epsilon \to 0, \text{ uniformly on compact sets.}$$

   Suppose also that

$$V_\epsilon(x) \geq V_\epsilon^\circ(x), \forall x \in \mathbf{R}, 0 < \epsilon \leq \epsilon_0$$

   where $V_\epsilon^\circ(x)$ is another family of measurable functions on $\mathbf{R}$, which is monotone decreasing in ϵ and $c(\epsilon) := \liminf_{|x|\to\infty} V_\epsilon^\circ(x) \to \infty$ as $\epsilon \to 0$. Then

$$\|\mathcal{J}_{V_\epsilon}^{-1} - \mathcal{J}_V^{-1}\| \to 0, \quad \epsilon \to 0$$

   where $\mathcal{J}_{V_\epsilon}, \mathcal{J}_V$ are the corresponding Schrödinger operator on $L^2(\mathbf{R})$ with the potential V_ϵ and V respectively.

2. $\|(\tilde{H}_{\epsilon,K} - \lambda)^{-1} - (\tilde{H}_0 - \lambda)^{-1}\| \to 0$ *as* $\epsilon \to 0$

Proof.

1. See [11] Page 113.

2. By Resolvent Identiy and the first part of the corollary, it suffices to check in this particular case

$$V(x) = 2a_0a_1x^m,\ V_\epsilon(x) = V(x) + \sum_{n=1}^{K} \tilde{H}_n \epsilon^{n\alpha}$$

where $\tilde{H}_n = (n+2)a_0a_1^{n+1}x^{nm+m} + (n-1)aa_0a_1^{n-2}x^{nm-2}$, satisfiy the conditions in the first part of the corollary. It is straightforward to see that

$$V_\epsilon(x) \to V(x) \text{ as } \epsilon \to 0, \text{ uniformly on compact sets.}$$

Now we need to construct $V_\epsilon^0(x)$. One can check

$$V_\epsilon(x) \geq \sigma \min(\epsilon^{-2+2\alpha_1+m\alpha_1}|x|^m, \epsilon^{-2+2\alpha_1}) = \sigma \min(|x|^m, \epsilon^{-2+2\alpha_1})$$

for some constant $\sigma > 0$. Hence if we take $V_\epsilon^0(x) = \sigma \min(|x|^m, \epsilon^{-2+2\alpha_1})$. Then the other conditions in the first part of the corollary is also satisfied. As a consequence we have $\|(\tilde{H}_{\epsilon,K} - \lambda)^{-1} - (\tilde{H}_0 - \lambda)^{-1}\| \to 0$ as $\epsilon \to 0$.

□

To move on we also need some information on the eigenfunctions $\phi_{\epsilon,K}$ of $\tilde{H}_{\epsilon,K}$, which will paly a role with the later construction of our testing function.

Lemma 4.2.7. *Let V be a positive $\mathbf{C}^\infty$ function on $\mathbf{R}$ and let $H = -\Delta + V$. Suppose that ψ is an eigenfunction of H. Then if $V(x) \geq s|x|^2 - t$ for some s and t, then for every $\epsilon > 0$, there is a D such that for all x we have*

$$|\psi(x)| \leq De^{-\frac{1}{2}\sqrt{s-\epsilon}|x|^2}$$

Proof. See [21] Page 252. □

Corollary 4.2.8. *Let* $\tilde{H}_{\epsilon,K} = \tilde{H}_0 + \sum_{n=1}^{K} \tilde{H}_n \epsilon^{n\alpha}$ *defined on* $\mathbf{R}$ *where*

$$\tilde{H}_0 = -\frac{d^2}{dy^2} + 2a_0a_1y^m, m \in 2\mathbf{Z}_+, \tilde{H}_n = (n+2)a_0a_1^{n+1}y^{nm+m} + (n-1)aa_0a_1^{n-2}y^{nm-2}$$

Let $\lambda(\tilde{H}_{\epsilon,K})$ *be the eigenvalue for* $\tilde{H}_{\epsilon,K}$ *with corresponding normalized eigenvector* $\phi_{\epsilon,K}$*.Then there exists a* D *such that for all* x *we have*

$$|\phi_{\epsilon,K}(x)| \leq De^{-\frac{1}{2}\sqrt{a_0a_1}|x|^2}$$

Proof. Direct application of **Lemma 4.2.7** with $s = 2a_0a_1$. □

With the eigenfunction $\phi_{\epsilon,K}$ we construct the following test function ϕ_K that will be used in proving our main result. The construction is stated in the following Lemma.

Lemma 4.2.9. *Let* $\phi_K = \phi_{\epsilon,K} \cdot f_\delta$*, where* $f_\delta(x) = \begin{cases} 1 & \text{if } x \in [-\frac{l_1}{\epsilon^{\alpha_1}} + \delta, \frac{l_2}{\epsilon^{\alpha_1}} - \delta] \\ 0 & \text{if } x \notin [-\frac{l_1}{\epsilon^{\alpha_1}}, \frac{l_2}{\epsilon^{\alpha_1}}] \end{cases}$
and $f_\delta(x) \in \mathbf{C}^\infty(\mathbf{R}), 1 \geq f_\delta(x) \geq 0$*. Then*

$$1 \geq ||\phi_K||^2_{L^2(\mathbf{I}_\epsilon)} \geq 1 - \frac{D^2}{\sqrt{a_0a_1}} e^{\delta\sqrt{a_0a_1}} (e^{-\sqrt{a_0a_1}\frac{l_1}{\epsilon^{\alpha_1}}} + e^{-\sqrt{a_0a_1}\frac{l_2}{\epsilon^{\alpha_1}}})$$

Proof. Let $\mathbf{I}_{\epsilon,\delta} = [-\frac{l_1}{\epsilon^{\alpha_1}} + \delta, \frac{l_2}{\epsilon^{\alpha_1}} - \delta]$, then

$$\begin{aligned}
||\phi_K||^2_{L^2(\mathbf{I}_\epsilon)} &= \int_{\mathbf{I}_\epsilon} f_\delta(x)^2 \phi_{\epsilon,K}(x)^2 dx \geq \int_{\mathbf{I}_{\epsilon,\delta}} f_\delta(x)^2 \phi_{\epsilon,K}(x)^2 dx = \int_{\mathbf{I}_{\epsilon,\delta}} \phi_{\epsilon,K}(x)^2 dx \\
&= \int_{\mathbf{R}} \phi_{\epsilon,K}(x)^2 dx - \int_{\mathbf{R}\setminus\mathbf{I}_{\epsilon,\delta}} \phi_{\epsilon,K}(x)^2 dx = 1 - \int_{\mathbf{R}\setminus\mathbf{I}_{\epsilon,\delta}} \phi_{\epsilon,K}(x)^2 dx \\
&\geq 1 - \int_{\mathbf{R}\setminus\mathbf{I}_{\epsilon,\delta}} [De^{-\frac{1}{2}\sqrt{a_0a_1}|x|^2}]^2 dx \\
&= 1 - \int_{-\infty}^{-\frac{l_1}{\epsilon^{\alpha_1}}+\delta} D^2 e^{-\sqrt{a_0a_1}|x|^2} dx - \int_{\frac{l_2}{\epsilon^{\alpha_1}}-\delta}^{\infty} D^2 e^{-\sqrt{a_0a_1}|x|^2} dx \\
&\geq 1 - \int_{-\infty}^{-\frac{l_1}{\epsilon^{\alpha_1}}+\delta} D^2 e^{-\sqrt{a_0a_1}|x|} dx - \int_{\frac{l_2}{\epsilon^{\alpha_1}}-\delta}^{\infty} D^2 e^{-\sqrt{a_0a_1}|x|} dx \\
&= 1 - \frac{D^2}{\sqrt{a_0a_1}} e^{\delta\sqrt{a_0a_1}} (e^{-\sqrt{a_0a_1}\frac{l_1}{\epsilon^{\alpha_1}}} + e^{-\sqrt{a_0a_1}\frac{l_2}{\epsilon^{\alpha_1}}})
\end{aligned}$$

On the other hand

$$||\phi_K||^2_{L^2(\mathbf{I}_\epsilon)} = \int_{\mathbf{I}_\epsilon} f_\delta(x)^2 \phi_{\epsilon,K}(x)^2 dx \leq \int_{\mathbf{I}_\epsilon} \phi_{\epsilon,K}(x)^2 dx \leq \int_{\mathbf{R}} \phi_{\epsilon,K}(x)^2 dx \leq 1$$

□

Remark 4.2.10. By our construction one sees directly that $\phi_K \in H^1_0(\mathbf{I}_\epsilon)$. Hence it would be a solid testing function for H_ϵ later.

Now we are ready to state the main results in the following theorem about the full eigenvalue asymptotics of $H_\epsilon = H_0 + \sum_{n=1}^\infty H_n \epsilon^{n\alpha}$ defined over $H^1_0(\mathbf{I}_\epsilon)$.

Theorem 4.2.11. *Let $\{\mu_j\}_{j=0}^\infty$ be the full set of eigenvalues of $\tilde{H}_0 = -\frac{d^2}{dy^2} + 2a_0a_1y^m$ defined on $H^1_0(\mathbf{R})$ with corresponding eigenfunctions $\{\psi_j\}_{j=0}^\infty$. Then the perturbed* *eigenvalue* λ *for* H_ϵ *around* μ_j *has asymptotic expansion given by*

$$\lambda \sim \mu_j + \sum_{n=1}^{\infty} q_n \epsilon^{n\alpha} \tag{4.3}$$

where

$$\begin{aligned} q_n &= \sum_{k=1}^{n} \frac{1}{2\pi \mathrm{i}} \mathbf{Tr} \int_\Gamma \left(\sum_{j_1+j_2+\cdots+j_k=n, j_i \in \mathbf{Z}^+} (-1)^k \lambda (H_0-\lambda)^{-1} H_{j_1} (H_0-\lambda)^{-1} H_{j_2} (H_0-\lambda)^{-1} \cdots H_{j_k} (H_0-\lambda)^{-1} \right) d\lambda \\ &= \sum_{k=1}^{n} \sum_{j_1+j_2+\cdots+j_k=n, j_i \in \mathbf{Z}^+} \sum_{s_1,s_2,\cdots,s_k=1}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{(-1)^k \lambda}{\mu_{s_1}-\lambda} \cdot \frac{a_{j_k s_1 s_2}}{\mu_{s_1}-\lambda} \cdot \frac{a_{j_{k-1} s_2 s_3}}{\mu_{s_2}-\lambda} \cdots \cdot \frac{a_{j_1 s_k s_1}}{\mu_{s_k}-\lambda} d\lambda \end{aligned}$$

with $\Gamma = \{\lambda : |\lambda - \mu_j| = \delta\}$ *any closed curve enclosing* μ_0 *and inside which* H_ϵ *has single eigenvalue and* $a_{nsk} = \langle H_n \psi_s, \psi_k \rangle$.

Proof. Let $\lambda^{(K)} = \mu_j + \sum_{n=1}^{K} q_n \epsilon^{n\alpha}$, $\lambda(\tilde{H}_{\epsilon,K})^{(K)} = \mu_j + \sum_{n=1}^{K} \tilde{q}_n \epsilon^{n\alpha}$. The first thing to observe is that $\lambda^{(K)} = \lambda(\tilde{H}_{\epsilon,K})^{(K)}$. Indeed we have

$$\begin{aligned} \lambda(H_{\epsilon,K})^{(K)} &= \mu_j + \sum_{n=1}^{K} \tilde{q}_n \epsilon^{n\alpha} \\ &= \mu_j + \sum_{n=1}^{K} \epsilon^{n\alpha} \sum_{k=1}^{n} \frac{1}{2\pi \mathrm{i}} \mathbf{Tr} \int_\Gamma \left(\sum_{j_1+j_2+\cdots+j_k=n, j_i \le K, j_i \in \mathbf{Z}^+} \frac{(-1)^k \lambda}{H_0-\lambda} H_{j_1} (H_0-\lambda)^{-1} \cdots H_{j_k} (H_0-\lambda)^{-1} \right) d\lambda \\ &= \mu_j + \sum_{n=1}^{K} q_n \epsilon^{n\alpha} \end{aligned}$$

Now by the self adjointness of H_ϵ and the fact that $\|(H_\epsilon - \lambda^{(K)})^{-1}\| = \frac{1}{dist(\lambda^{(K)}, spec(H_\epsilon))}$, also with the help of **Remark 4.2.10** and **Lemma 4.2.9**, it suffices to show

$$\|(H_\epsilon - \lambda^{(K)}) \phi_K\| = O(\epsilon^{(K+1)\alpha}).$$

Using the observation $\lambda^{(K)} = \lambda(\tilde{H}_{\epsilon,K})^{(K)}$, we have

$$
\begin{aligned}
& ||(H_\epsilon - \lambda^{(K)})\phi_K|| \\
=& ||H_\epsilon \phi_K - \lambda(\tilde{H}_{\epsilon,K})^{(K)}\phi_K|| \\
=& ||\left(\tilde{H}_{\epsilon,K} + (H_\epsilon - \tilde{H}_{\epsilon,K})\right)\phi_K - \lambda(\tilde{H}_{\epsilon,K})^{(K)}\phi_K| \\
=& ||\left(H_\epsilon - \tilde{H}_{\epsilon,K}\right)\phi_K + \left(\tilde{H}_{\epsilon,K} - \lambda(\tilde{H}_{\epsilon,K})^{(K)}\right)\phi_K|| \\
=& ||\left(H_\epsilon - \tilde{H}_{\epsilon,K}\right)\phi_K + \left(\lambda(\tilde{H}_{\epsilon,K}) - \lambda(\tilde{H}_{\epsilon,K})^{(K)}\right)\phi_K - \phi_{\epsilon,K} f''_\delta - 2\phi'_{\epsilon,K} f'_\delta|| \\
\leq& O(\epsilon^{(K+1)\alpha})||\phi_K||_{L^2(\mathbf{I}_\epsilon)} + ||(H_\epsilon - \tilde{H}_{\epsilon,K})\phi_K|| + ||\phi_{\epsilon,K} f''_\delta|| + ||2\phi'_{\epsilon,K} f'_\delta||
\end{aligned}
$$

Notice now that $||\phi_{\epsilon,K} f''_\delta|| + ||2\phi'_{\epsilon,K} f'_\delta|| = O(\epsilon^{(K+1)\alpha})$ when $\delta \to 0$ by absolute continuity and the fact that f'_δ, f''_δ are bounded and supported on $[-\frac{l_1}{\epsilon_1^\alpha}, -\frac{l_1}{\epsilon_1^\alpha} + \delta] \cup [\frac{l_2}{\epsilon_1^\alpha} - \delta, \frac{l_2}{\epsilon_1^\alpha}]$. So to prove the theorem, it suffices to show $||(H_\epsilon - \tilde{H}_{\epsilon,K})\phi_K|| = O(\epsilon^{(K+1)\alpha})$.

Noticing ϕ_K is supported on $\mathbf{I}_\epsilon$, $||(H_\epsilon - \tilde{H}_{\epsilon,K})\phi_K|| = \|(H_\epsilon - H_{\epsilon,K})\phi_K\|$, where $H_{\epsilon,K}$ is just the restriction of $\tilde{H}_{\epsilon,K}$ to $\mathbf{I}_\epsilon$, which is also the truncation of the taylor expansion of the potential of H_ϵ up to the first K terms. By analyticity of the potential of H_ϵ, clearly one has $|(H_\epsilon - H_{\epsilon,K})| \leq O(\epsilon^{(K+1)\alpha} y^{(K+2)m})$. And by the exponetial decaying of $\phi_{\epsilon,K}$ from **Corollary 4.2.8** and the fact that $|\phi_K| \leq |\phi_{\epsilon,K}|$, we obtain $||(H_\epsilon - \tilde{H}_{\epsilon,K})\phi_K|| = O(\epsilon^{(K+1)\alpha})$.

For readers who prefer explicit fomulas we also give an alternative proof of the last argument involving explicit calculations as follows: In fact, because H_ϵ is defined on $\mathbf{I}_\epsilon = [-\frac{l_1}{\epsilon^{\alpha_1}}, \frac{l_2}{\epsilon^{\alpha_1}}]$, we have that

$$a_1\epsilon^\alpha y^m \le \max\{c\frac{l_1^m}{M}, c\frac{l_2^m}{M}\}\hat{=}q < 1 \quad \text{since} \quad h(x) > 0$$

And notice

$$\begin{aligned}
&\sum_{n=K+1}^{\infty} H_n\epsilon^{n\alpha} \\
=&\sum_{n=K+1}^{\infty}[a_0\frac{1}{\epsilon^\alpha}(n+2)(a_1y^m\epsilon^\alpha)^{n+1} + aa_0\epsilon^{2\alpha}y^{2m-2}(n-1)(a_1y^m\epsilon^\alpha)^{n-2}] \\
=&a_0\frac{1}{\epsilon^\alpha}\frac{(K+3)(a_1y^m\epsilon^\alpha)^{K+2} - (K+2)(a_1y^m\epsilon^\alpha)^{K+3}}{(1-a_1y^m\epsilon^\alpha)^2} + aa_0\epsilon^{2\alpha}y^{2m-2}\frac{K(a_1y^m\epsilon^\alpha)^{K-1} + (1-K)(a_1y^m\epsilon^\alpha)^K}{(1-a_1y^m\epsilon^\alpha)^2} \\
\le&a_0\frac{1}{(1-q)^2}[(K+3)(a_1y^m)^{K+2}\epsilon^{(K+1)\alpha} - (K+2)(a_1y^m)^{K+3}\epsilon^{(K+2)\alpha}] \\
&+ aa_0\frac{1}{(1-q)^2}[K(a_1y^m)^{K-1}y^{2m-2}\epsilon^{(K+1)\alpha} - (K-1)y^{2m-2}(a_1y^m)^K\epsilon^{(K+2)\alpha}] \\
\le&\epsilon^{(K+1)\alpha}[\frac{a_0}{(1-q)^2}(K+3)(a_1y^m)^{K+2} + \frac{aa_0}{(1-q)^2}Ka_1^{K-1}y^{(K+1)m-2}] \\
\le&\epsilon^{(K+1)\alpha}Cy^{(K+2)m}
\end{aligned}$$

where C is some constant depending only on K, a, a_0, a_1 and q. We also notice from **Corollary 4.2.8** that

$$||y^{(K+2)m}\phi_K||^2_{L^2(\mathbf{I}_\epsilon)} = \int_{\mathbf{I}_\epsilon} y^{2(K+2)m}\phi_K^2 \le \int_{\mathbf{R}} y^{2(K+2)m}\phi_{\epsilon,K}^2 \le \int_{\mathbf{R}} y^{2(K+2)m}D^2e^{-\sqrt{a_0a_1}|y|^2} < \infty$$

Most importantly the bound here is not involving ϵ. Thus

$$||(H_\epsilon - H_{\epsilon,K})\phi_K| = ||\sum_{n=K+1}^{\infty} H_n\epsilon^{n\alpha}\phi_K|| \le ||\epsilon^{(K+1)\alpha}Cy^{(K+2)m}\phi_K|| = O(\epsilon^{(K+1)\alpha})$$

In conclusion we just showed that $\lambda \sim \mu_j + \sum_{n=1}^{\infty} q_n\epsilon^{n\alpha}$.

$\square$

4.3 Eigenvalue Asymptotics of A_ϵ in the general case

We now move on to the discussion of the general case. In this case $h(x)$ is a positive analytic function having 0 the only point where it achieves its global maximum M. With such assumptions on $h(x)$, one easily see $h(x) = M - c(x)x^m$ for some even integer m and some positive analytic function $c(x)$ with $c(0) = c_0 \neq 0$.

For later discussion let's recall $c(x) = \sum_{n=0}^{\infty} c_n x^n$ is analytic. Because of that we also have the following analytic functions

$$c(x)^2 = \sum_{n=0}^{\infty} d_n x^n \qquad c'(x) = \sum_{n=0}^{\infty} e_n x^n$$

$$c'(x)^2 = \sum_{n=0}^{\infty} g_n x^n \qquad c(x)c'(x) = \sum_{n=0}^{\infty} f_n x^n$$

$$\left(\frac{c(x)}{M}\right)^{n+1} = \sum_{k=0}^{\infty} \alpha_{k,n} x^k \qquad \left(\frac{c(x)}{M}\right)^{n-2} = \sum_{k=0}^{\infty} \beta_{k,n} x^k$$

We adopt the convention that $t_j = 0$ for all the negative indicies with t_j being any of those coefficients in the expansions above. And for simplification of the notations, all the notations will be understood within the context of this section and should not be viewed as conflicted with the same notations in other sections.

To start we have a similar result as **Lemma 4.2.1** for the particular case by introducing a proper scaling and talyor expanding the potential. In particular we have the following:

Lemma 4.3.1. *Let* $x = \epsilon^{\alpha_1} y$ *where* $\alpha_1 = \frac{2}{m+2}, y \in \mathbf{I}_\epsilon = [-\frac{l_1}{\epsilon^{\alpha_1}}, \frac{l_2}{\epsilon^{\alpha_1}}]$, *then*

$$A_\epsilon \text{ is unitarily equivalent to } H_\epsilon = H_0 + \sum_{n=1}^{\infty} H_n \epsilon^{n\alpha_1}$$

with $a_0 = \frac{\pi^2}{M^2}$ *and* $H_0 = -\frac{d^2}{dy^2} + 2a_0 a_1 y^m, H_n$ *is a polynomial of degree* $n+m$. *More precisely*

$$H_n = \frac{2a_0 c_n}{M} y^{n+m} + \sum_{k+sm=n, k\in\mathbf{N}, s\geq 1} \left[(s+2) a_0 \alpha_{k,s} y^{n+m} + \sum_{i+j=k, i,j\in\mathbf{N}} \beta_{i,s} y^i \gamma_j \right]$$

where $\gamma_j = A_1 d_j y^j + A_2 f_{j-1} y^{j-1} + A_3 g_{j-2} y^{j-2}$ *and* $A_1 = (\frac{\pi^2}{3} + \frac{1}{4})\frac{m^2}{\pi^2}(n-1)a_0, A_2 = \frac{2m}{M^2}(\frac{\pi^2}{3} + \frac{1}{4})(n-1), A_3 = (\frac{\pi^2}{3} + \frac{1}{4})(n-1)\frac{1}{M^2}$ *are some pure constants.*

Proof. The proof is similar as **Lemma 4.2.1**. And it is contained in Appendix A. □

Remark 4.3.2. Comparing **Lemma 4.3.1** and **Lemma 4.2.1**, we see the main difference between the general case and the particular case is that the H_n is a polynomial of degree $n+m$ in the general case, which could be genuinely of odd degree. While in the particular case one always have an even degree $(n+1)m$.

Because of the **Remark 4.3.2**, one can not follow the steps as in the particular case by considering $\tilde{H}_{\epsilon,K} = \tilde{H}_0 + \sum_{n=1}^{K} \tilde{H}_n \epsilon^{n\alpha}$, where $\tilde{H}_0 = H_0, \tilde{H}_n = H_n$ except they are defined on $\mathbf{R}$ instead of $\mathbf{I}_\epsilon$. As in this case the spectrum of $\tilde{H}_{\epsilon,K}$ has a complicated behavior, one might only have resonants for $\tilde{H}_{\epsilon,K}$ without any eigenvalues for ϵ real. Thus we need a proper substituation of $\tilde{H}_{\epsilon,K}$ in the particular case.

To proceed we make the following constructions. Let $\tilde{H}_0 = -\frac{d^2}{dy^2} + 2a_0a_1y^m$ be the same as H_0 except that it is defined over $H_0^1(\mathbf{R})$. Let $\tilde{H}_n = H_n \cdot \chi_{\mathbf{I}_\epsilon}(y)$ where $\chi_{\mathbf{I}_\epsilon}(y)$ is the charateristic function over $\mathbf{I}_\epsilon = [-\frac{l_1}{\epsilon^{\alpha_1}}, \frac{l_2}{\epsilon^{\alpha_1}}]$.

Lemma 4.3.3. *We have the following*

1. $H_\epsilon = \tilde{H}_0 \cdot \chi_{\mathbf{I}_\epsilon}(y) + \sum_{n=1}^{\infty} \tilde{H}_n \epsilon^{n\alpha_1}$

2. $\exists$ *a pure consant* T *such that* $\tilde{H}_{\epsilon,K} = \tilde{H}_0 + \sum_{n=1}^{K} \tilde{H}_n \epsilon^{n\alpha_1} + T\epsilon^{2\alpha_1 - 2}$ *is a positive operator and the potential of* $\tilde{H}_{\epsilon,K}$ *goes to* ∞ *as* $|y| \to \infty$*. Moreover*

$$\tilde{H}_{\epsilon,K} \geq -\frac{d^2}{dy^2} + 2a_0a_1y^m$$

Proof. The proof goes as

1. It follows from our constructions of $\tilde{H}_0, \tilde{H}_n$.

2. Let $V(x) = \frac{\pi^2}{\epsilon^2 h^2(x)} - \frac{\pi^2}{\epsilon^2 M^2} + \left(\frac{\pi^2}{3} + \frac{1}{4}\right)\frac{h'(x)^2}{h(x)^2}$. By analyticity we assume

$$\frac{\pi^2}{h^2(x)} - \frac{\pi^2}{M^2} = \sum_{n=0}^{\infty} h_n x^n, \quad \left(\frac{\pi^2}{3} + \frac{1}{4}\right)\frac{h'^2(x)}{h^2(x)} = \sum_{n=0}^{\infty} t_n x^n.$$

   Thus

$$V(x) = \frac{1}{\epsilon^2}\sum_{n=0}^{\infty} h_n x^n + \sum_{n=0}^{\infty} t_n x^n$$

Notice $(\tilde{H}_0 + \frac{d^2}{dy^2}) \cdot \chi_{\mathbf{I}_\epsilon}(y) + \sum_{n=1}^{K} \tilde{H}_n \epsilon^{n\alpha_1}$ comes from the taylor expansion of $\epsilon^{2\alpha_1} V(\epsilon^{\alpha_1} y)$. In particular

$$(\tilde{H}_0 + \frac{d^2}{dy^2}) \cdot \chi_{\mathbf{I}_\epsilon}(y) + \sum_{n=1}^{K} \tilde{H}_n \epsilon^{n\alpha_1} = \epsilon^{2\alpha_1 - 2} \sum_{n=0}^{K+m} h_n (\epsilon^{\alpha_1} y)^n + \epsilon^{2\alpha_1} \sum_{n=0}^{K-2} t_n (\epsilon^{\alpha_1} y)^n$$

Here we adopt the convention that $t_j = 0$ for all the negative indicies j.

Hence by the error control of taylor expansion we have

$$\begin{aligned} &\left| \epsilon^{2\alpha_1} V(\epsilon^{\alpha_1} y) - \left((\frac{d^2}{dy^2} + \tilde{H}_0) \cdot \chi_{\mathbf{I}_\epsilon}(y) + \sum_{n=1}^{K} \tilde{H}_n \epsilon^{n\alpha_1} \right) \right| \\ \leq &\epsilon^{2\alpha_1 - 2} |h_{K+m+1}| \cdot \left| (\epsilon^{\alpha_1} y)^{K+m+1} \right| + \epsilon^{2\alpha_1} |t_{K-1}| \cdot \left| (\epsilon^{\alpha_1} y)^{K-1} \right| \end{aligned}$$

Combining with the fact that $y \in \mathbf{I}_\epsilon = [-\frac{l_1}{\epsilon^{\alpha_1}}, \frac{l_2}{\epsilon^{\alpha_2}}]$ and $|V(x)| \leq T_1$, we have

$$\begin{aligned} &\left| \left((\frac{d^2}{dy^2} + \tilde{H}_0) \cdot \chi_{\mathbf{I}_\epsilon}(y) + \sum_{n=1}^{K} \tilde{H}_n \epsilon^{n\alpha_1} \right) \right| \\ \leq &\left| \epsilon^{2\alpha_1} V(\epsilon^{\alpha_1} y) \right| + \epsilon^{2\alpha_1 - 2} |h_{K+m+1}| \cdot \left| (\epsilon^{\alpha_1} y)^{K+m+1} \right| + \epsilon^{2\alpha_1} |t_{K-1}| \cdot \left| (\epsilon^{\alpha_1} y)^{K-1} \right| \\ \leq &\epsilon^{2\alpha_1} T_1 + \epsilon^{2\alpha_1 - 2} |h_{K+m+1}| \max\{l_1, l_2\}^{K+m+1} + \epsilon^{2\alpha_1} |t_{K-1}| \max\{l_1, l_2\}^{K-1} \\ \leq &T \epsilon^{2\alpha_1 - 2} \end{aligned}$$

for some constant T determined by $T_1, |h_{K+m+1}|, |t_{K-1}|, \max\{l_1, l_2\}^K$.

In particular the last bound implied that $\tilde{H}_{\epsilon,K} = \tilde{H}_0 + \sum_{n=1}^{K} \tilde{H}_n \epsilon^{n\alpha_1} + T\epsilon^{2\alpha_1 - 2}$ is a positive operator on $H_0^1(\mathbf{R})$. And the potential of $\tilde{H}_{\epsilon,K}$ goes to ∞ as $|y| \to \infty$. In particular we have $\tilde{H}_{\epsilon,K} \geq -\frac{d^2}{dy^2} + 2a_0 a_1 y^m$ on $\mathbf{R} \backslash \mathbf{I}_\epsilon$.

Noticing $\alpha_1 = \frac{2}{m+2}$ and over $\mathbf{I}_\epsilon$,

$$|2a_0a_1y^m| \leq 2a_0a_1 \max\{l_1, l_2\}^m \epsilon^{-m\alpha_1} = 2a_0a_1 \max\{l_1, l_2\}^m \epsilon^{2\alpha_1 - 2}$$

So one can refine T to make $\tilde{H}_{\epsilon,K} \geq -\frac{d^2}{dy^2} + 2a_0a_1y^m$ on $\mathbf{R}$.

□

Remark 4.3.4. The $\tilde{H}_{\epsilon,K}$ constructed above in **Lemma 4.3.3** now is going to play a similar role as in the particular case. One thing different from the particular case is that there is a constant shiftting term $T\epsilon^{2\alpha_1-2}$ involved, which is necessary to make sure we are dealing with positve operator and hence have a nice spectral theory. On the other hand it is just a constant shiftting and thus no extra difficulty introduced.

Now the analysis is parallel as what we did in the particular case. In particular we have

Lemma 4.3.5. *Let $\{\mu_j\}_{j=0}^{\infty}$ be the full set of eigenvalues of $\tilde{H}_0$ defined on $H_0^1(\mathbf{R})$ with corresponding eigenfunctions $\{\psi_j\}_{j=0}^{\infty}$. Let $\lambda(\tilde{H}_{\epsilon,K})$ be the eigenvalue for $\tilde{H}_{\epsilon,K} = \tilde{H}_0 + \sum_{n=1}^{K} \tilde{H}_n \epsilon^{n\alpha_1} + T\epsilon^{2\alpha_1-2}$ defined on $H_0^1(\mathbf{R})$ near μ_j with corresponding normalized eigenvector $\phi_{\epsilon,K}$. Then*

$$\lambda(\tilde{H}_{\epsilon,K}) - T\epsilon^{2\alpha_1-2} \sim \mu_j + \sum_{n=1}^{\infty} \epsilon^{n\alpha} \tilde{q}_n \tag{4.4}$$

where

$$\tilde{q}_n = \sum_{k=1}^{n} \frac{1}{2\pi \mathrm{i}} \mathbf{Tr} \int_{\Gamma} \left(\sum_{j_1+j_2+\cdots+j_k=n, j_i \le K, j_i \in \mathbf{Z}^+} (-1)^k \lambda (\tilde{H}_0 - \lambda)^{-1} H_{j_1} (\tilde{H}_0 - \lambda)^{-1} \cdots H_{j_k} (\tilde{H}_0 - \lambda)^{-1} \right) d\lambda$$

and $\Gamma = \{\lambda : |\lambda - \mu_j| = \delta\}$ *any closed curve enclosing* μ_j *and inside which* $\tilde{H}_{\epsilon,K}$ *has single eigenvalue.*

Proof. The proof is almost identical as **Lemma 4.2.4** cxcept we are now thinking $\tilde{H}_{\epsilon,K}$ as perturbing $\tilde{H}_0 + T\epsilon^{2\alpha_1-2}$, which has the spectrum as $\tilde{H}_0$ but shiftted with $T\epsilon^{2\alpha_1-2}$. To finish the proof we also need to show as $\epsilon \to 0$, we have

$$\|(\tilde{H}_{\epsilon,K} - \lambda)^{-1} - (\tilde{H}_0 + T\epsilon^{2\alpha_1-2} - \lambda)^{-1}\| \to 0$$

which follows from **Proposition 4.2.5** and **Corollary 4.2.6**. □

Similarly as the particular case in previous section we have the following construction of testing functions.

Lemma 4.3.6. *Let* $\tilde{H}_{\epsilon,K} = \tilde{H}_0 + \sum_{n=1}^{K} \tilde{H}_n \epsilon^{n\alpha_1} + T\epsilon^{2\alpha_1-2}$ *defined on* $\mathbf{R}$*. Let* $\lambda(\tilde{H}_{\epsilon,K})$ *be the eigenvalues for* $\tilde{H}_{\epsilon,K}$ *with corresponding eigenvector* $\phi_{\epsilon,K}$*. Then there exists a constant* D *such that for all* y *we have*

$$|\phi_{\epsilon,K}(y)| \le De^{-\frac{1}{2}\sqrt{a_0 a_1}|y|^2}$$

Proof. Based on part 2 of **Lemma 4.3.3** this is a direct application of **Lemma 4.2.7** with $s = 2a_0a_1$. □

Lemma 4.3.7. *Let $\phi_K = \phi_{\epsilon,K} \cdot f_\delta$, where* $f_\delta(x) = \begin{cases} 1 & \textit{if } x \in [-\frac{l_1}{\epsilon^{\alpha_1}} + \delta, \frac{l_2}{\epsilon^{\alpha_1}} - \delta] \\ \\ 0 & \textit{if } x \notin [-\frac{l_1}{\epsilon^{\alpha_1}}, \frac{l_2}{\epsilon^{\alpha_1}}] \end{cases}$
and $f_\delta(x) \in \mathbf{C}^\infty(\mathbf{R}), 1 \geq f_\delta(x) \geq 0$. Then

$$1 \geq ||\phi_K||^2_{L^2(\mathbf{I}_\epsilon)} \geq 1 - \frac{D^2}{\sqrt{a_0 a_1}} e^{\delta\sqrt{a_0 a_1}} (e^{-\sqrt{a_0 a_1}\frac{l_1}{\epsilon^{\alpha_1}}} + e^{-\sqrt{a_0 a_1}\frac{l_2}{\epsilon^{\alpha_1}}})$$

Proof. Let $\mathbf{I}_{\epsilon,\delta} = [-\frac{l_1}{\epsilon^{\alpha_1}} + \delta, \frac{l_2}{\epsilon^{\alpha_1}} - \delta]$, then

$$\begin{aligned} ||\phi_K||^2_{L^2(\mathbf{I}_\epsilon)} &= \int_{\mathbf{I}_\epsilon} f_\delta(x)^2 \phi_{\epsilon,K}(x)^2 dx \geq \int_{\mathbf{I}_{\epsilon,\delta}} f_\delta(x)^2 \phi_{\epsilon,K}(x)^2 dx = \int_{\mathbf{I}_{\epsilon,\delta}} \phi_{\epsilon,K}(x)^2 dx \\ &= \int_{\mathbf{R}} \phi_{\epsilon,K}(x)^2 dx - \int_{\mathbf{R}\setminus\mathbf{I}_{\epsilon,\delta}} \phi_{\epsilon,K}(x)^2 dx = 1 - \int_{\mathbf{R}\setminus\mathbf{I}_{\epsilon,\delta}} \phi_{\epsilon,K}(x)^2 dx \\ &\geq 1 - \int_{\mathbf{R}\setminus\mathbf{I}_{\epsilon,\delta}} [D e^{-\frac{1}{2}\sqrt{a_0 a_1}|x|^2}]^2 dx \\ &= 1 - \int_{-\infty}^{-\frac{l_1}{\epsilon^{\alpha_1}}+\delta} D^2 e^{-\sqrt{a_0 a_1}|x|^2} dx - \int_{\frac{l_2}{\epsilon^{\alpha_1}}-\delta}^{\infty} D^2 e^{-\sqrt{a_0 a_1}|x|^2} dx \\ &\geq 1 - \int_{-\infty}^{-\frac{l_1}{\epsilon^{\alpha_1}}+\delta} D^2 e^{-\sqrt{a_0 a_1}|x|} dx - \int_{\frac{l_2}{\epsilon^{\alpha_1}}-\delta}^{\infty} D^2 e^{-\sqrt{a_0 a_1}|x|} dx \\ &= 1 - \frac{D^2}{\sqrt{a_0 a_1}} e^{\delta\sqrt{a_0 a_1}} (e^{-\sqrt{a_0 a_1}\frac{l_1}{\epsilon^{\alpha_1}}} + e^{-\sqrt{a_0 a_1}\frac{l_2}{\epsilon^{\alpha_1}}}) \end{aligned}$$

On the other hand

$$||\phi_K||^2_{L^2(\mathbf{I}_\epsilon)} = \int_{\mathbf{I}_\epsilon} f_\delta(x)^2 \phi_{\epsilon,K}(x)^2 dx \leq \int_{\mathbf{I}_\epsilon} \phi_{\epsilon,K}(x)^2 dx \leq \int_{\mathbf{R}} \phi_{\epsilon,K}(x)^2 dx \leq 1$$

□

With all these preparations we now state and prove the eigenvalue asymptotics of A_ϵ in the general case.

Theorem 4.3.8. *Let $\{\mu_j\}_{j=0}^{\infty}$ be the full set of eigenvalues of $\tilde{H}_0 = -\frac{d^2}{dy^2} + 2a_0a_1y^m$ defined on $H_0^1(\mathbf{R})$ with corresponding eigenfunctions $\{\psi_j\}_{j=0}^{\infty}$. Then the perturbed eigenvalue λ for H_ϵ around μ_j has asymptotic expansion given by*

$$\lambda \sim \mu_j + \sum_{n=1}^{\infty} q_n \epsilon^{n\alpha} \tag{4.5}$$

where

$$\begin{aligned} q_n &= \sum_{k=1}^{n} \frac{1}{2\pi \mathrm{i}} \mathbf{Tr} \int_\Gamma \left(\sum_{j_1+j_2+\cdots+j_k=n, j_i \in \mathbf{Z}^+} (-1)^k \lambda (H_0-\lambda)^{-1} H_{j_1} (H_0-\lambda)^{-1} H_{j_2} (H_0-\lambda)^{-1} \cdots H_{j_k} (H_0-\lambda)^{-1} \right) d\lambda \\ &= \sum_{k=1}^{n} \sum_{j_1+j_2+\cdots+j_k=n, j_i \in \mathbf{Z}^+} \sum_{s_1, s_2, \cdots, s_k=1}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{(-1)^k \lambda}{\mu_{s_1}-\lambda} \cdot \frac{a_{j_k s_1 s_2}}{\mu_{s_1}-\lambda} \cdot \frac{a_{j_{k-1} s_2 s_3}}{\mu_{s_2}-\lambda} \cdots \cdot \frac{a_{j_1 s_k s_1}}{\mu_{s_k}-\lambda} d\lambda \end{aligned}$$

with $\Gamma = \{\lambda : |\lambda - \mu_j| = \delta\}$ any closed curve enclosing μ_0 and inside which H_ϵ has single eigenvalue and $a_{nsk} = \langle H_n \psi_s, \psi_k \rangle$.

Proof. Let $\lambda^{(K)} = \mu_j + \sum_{n=1}^{K} q_n \epsilon^{n\alpha}$, $\lambda(\tilde{H}_{\epsilon,K})^{(K)} = \mu_j + \sum_{n=1}^{K} \tilde{q}_n \epsilon^{n\alpha}$. The first thing to observe is that $\lambda^{(K)} = \lambda(\tilde{H}_{\epsilon,K})^{(K)}$. Indeed we have

$$\begin{aligned} \lambda(H_{\epsilon,K})^{(K)} &= \mu_j + \sum_{n=1}^{K} \tilde{q}_n \epsilon^{n\alpha} \\ &= \mu_j + \sum_{n=1}^{K} \epsilon^{n\alpha} \sum_{k=1}^{n} \frac{1}{2\pi \mathrm{i}} \mathbf{Tr} \int_\Gamma \left(\sum_{j_1+j_2+\cdots+j_k=n, j_i \le K, j_i \in \mathbf{Z}^+} \frac{(-1)^k \lambda}{H_0-\lambda} H_{j_1} (H_0-\lambda)^{-1} \cdots H_{j_k} (H_0-\lambda)^{-1} \right) d\lambda \\ &= \mu_j + \sum_{n=1}^{K} q_n \epsilon^{n\alpha} \end{aligned}$$

Now by the self adjointness of H_ϵ and the fact that $\|(H_\epsilon - \lambda^{(K)})^{-1}\| = \frac{1}{dist(\lambda^{(K)}, spec(H_\epsilon))}$, also with the help of **Remark 4.2.10** and **Lemma 4.3.7**, it suffices to show

$$\|(H_\epsilon - \lambda^{(K)})\phi_K\| = O(\epsilon^{(K+1)\alpha}).$$

Using the observation $\lambda^{(K)} = \lambda(\tilde{H}_{\epsilon,K})^{(K)}$, we have

$$
\begin{aligned}
&||(H_\epsilon - \lambda^{(K)})\phi_K|| \\
=&||(H_\epsilon + T\epsilon^{2\alpha_1-2})\phi_K - \lambda(\tilde{H}_{\epsilon,K})^{(K)}\phi_K|| \\
=&|| \left(\tilde{H}_{\epsilon,K} + (H_\epsilon + T\epsilon^{2\alpha_1-2} - \tilde{H}_{\epsilon,K})\right)\phi_K - \lambda(\tilde{H}_{\epsilon,K})^{(K)}\phi_K| \\
=&|| \left(H_\epsilon + T\epsilon^{2\alpha_1-2} - \tilde{H}_{\epsilon,K}\right)\phi_K + \left(\tilde{H}_{\epsilon,K} - \lambda(\tilde{H}_{\epsilon,K})^{(K)}\right)\phi_K|| \\
=&|| \left(H_\epsilon + T\epsilon^{2\alpha_1-2} - \tilde{H}_{\epsilon,K}\right)\phi_K + \left(\lambda(\tilde{H}_{\epsilon,K}) - \lambda(\tilde{H}_{\epsilon,K})^{(K)}\right)\phi_K - \phi_{\epsilon,K}f''_\delta - 2\phi'_{\epsilon,K}f'_\delta|| \\
\leq&O(\epsilon^{(K+1)\alpha_1})||\phi_K||_{L^2(\mathbf{I}_\epsilon)} + ||(H_\epsilon + T\epsilon^{2\alpha_1-2} - \tilde{H}_{\epsilon,K})\phi_K|| + ||\phi_{\epsilon,K}f''_\delta|| + ||2\phi'_{\epsilon,K}f'_\delta||
\end{aligned}
$$

Notice now that $||\phi_{\epsilon,K}f''_\delta|| + ||2\phi'_{\epsilon,K}f'_\delta|| \to 0$ when $\delta \to 0$ by absolute continuity and the fact that f'_δ, f''_δ are bounded and supported on $[-\frac{l_1}{\epsilon_1^\alpha}, -\frac{l_1}{\epsilon_1^\alpha} + \delta] \cup [\frac{l_2}{\epsilon_1^\alpha} - \delta, \frac{l_2}{\epsilon_1^\alpha}]$. So to prove the theorem, it suffices to show $||(H_\epsilon + T\epsilon^{2\alpha_1-2} - \tilde{H}_{\epsilon,K})\phi_K|| = O(\epsilon^{(K+1)\alpha_1})$.

Noticing ϕ_K is supported on $\mathbf{I}_\epsilon$, $||(H_\epsilon + T\epsilon^{2\alpha_1-2} - \tilde{H}_{\epsilon,K})\phi_K|| = \|(H_\epsilon - H_{\epsilon,K})\phi_K\|$, where $H_{\epsilon,K}$ is just the restriction of $\tilde{H}_{\epsilon,K} - T\epsilon^{2\alpha_1-2}$ to $\mathbf{I}_\epsilon$, which is also the truncation of the taylor expansion of the potential of H_ϵ up to the first K terms. By analyticity of the potential of H_ϵ, clearly one has $|(H_\epsilon - H_{\epsilon,K})| \leq O(\epsilon^{(K+1)\alpha_1}y^{(K+m+1)})$. And by the exponetial decaying of $\phi_{\epsilon,K}$ from **Lemma 4.3.6** and the fact that $|\phi_K| \leq |\phi_{\epsilon,K}|$, we obtain $||(H_\epsilon - \tilde{H}_{\epsilon,K})\phi_K|| = O(\epsilon^{(K+1)\alpha_1})$. $\square$

CHAPTER 5

Study of the Difference between A_{11} and Δ_ϵ

In the previous sections we studied in details about the asymptotics of the eigenvalues of A_{11}. More precisely we showed $\epsilon^{2\alpha_1}\left(A_{11} - \frac{\pi^2}{M^2\epsilon^2}\right)$ has eigenvalue asymptotics

$$\nu \sim \mu_0 + \sum_{n=1}^{\infty} q_n \epsilon^{n\alpha_1}.$$

To go back to the full asymptotics of Dirichelt Laplacian, we need to figure out the difference $\tilde{\lambda} = \Lambda_\epsilon - \lambda$ between the eigenvalues Λ_ϵ of the original Dirichlet Laplacian Δ_ϵ and the eigenvalues λ of the model operator A_{11}.To fix the notations we let λ be the eigenvalues of A_{11} with corresponding normalized eigenfunction ϕ. Then

$$A_{11}\phi = \lambda\phi.$$

The study of the difference $\tilde{\lambda}$ proceeds in two steps. In **Section 5.1** we will derive the equation that is satisfied by $\tilde{\lambda}$. Then in **Section 5.2** we will develop an iterative scheme for solving the equation.

5.1 Equation for the Difference $\tilde{\lambda}$

Recall from Theorem A

$$\begin{aligned} A_{11}u_1 + A_{12}u_2 &= (\lambda + \tilde{\lambda})u_1 \\ A_{21}u_1 + A_{22}u_2 &= (\lambda + \tilde{\lambda})u_2 \end{aligned} \tag{5.1}$$

where $u_1 = Pu, u_2 = Qu$. The two functions u_1 and u_2 are actually connected as shown in the following Lemma.

Lemma 5.1.1.

1. *$A_{22} - \lambda$ is invertible. Moreover $\|(A_{22} - \lambda)^{-1}\| \leq C\epsilon^2$ for some pure constant C.*

2. *$\lim_{\epsilon \to 0} \epsilon^{2\alpha_1}\tilde{\lambda} = 0$.*

3. *$\|\tilde{\lambda}(A_{22} - \lambda)^{-1}\| < 1$ as $\epsilon \to 0$. Moreover $I - \tilde{\lambda}(A_{22} - \lambda)^{-1}$ is invertible and one has the Neumann Expansion:*

$$\left[I - \tilde{\lambda}(A_{22} - \lambda)^{-1}\right]^{-1} = \sum_{n=0}^{\infty} \tilde{\lambda}^n (A_{22} - \lambda)^{-n}$$

4. *$u_1 = Pu, u_2 = Qu$ satisfies the following*

$$u_2 = -\sum_{n=0}^{\infty} \tilde{\lambda}^n (A_{22} - \lambda)^{-(n+1)} A_{21}u_1 \tag{5.2}$$

Proof. From (5.1) we have

$$u_2 = (A_{22} - \lambda)^{-1}(\tilde{\lambda}u_2 - A_{21}u_1) \tag{5.3}$$

Using (5.3) iteratively, we have

$$u_2 = -[I - \tilde{\lambda}(A_{22} - \lambda)^{-1}]^{-1}[(A_{22} - \lambda)^{-1} A_{21} u_1]$$

Inside the proof there are two things that should be explained more, namely the invertibility of $(A_{22} - \lambda)$ and the invertibility of $I - \tilde{\lambda}(A_{22} - \lambda)^{-1}$.

$(A_{22} - \lambda)$ is invertible because of the following observations. First we already see that $\lambda \sim \frac{\pi^2}{M^2\epsilon^2}$. Using variational characterization (Energy Estimate) one easily see that the spectrum of $A_{22} = Q\Delta_\epsilon Q$ is bounded below by $\frac{4\pi^2}{M^2\epsilon^2}$. More precisely one can show $\frac{\langle A_{22}v, v\rangle}{\langle v, v\rangle} \geq \frac{4\pi^2}{\epsilon^2 M^2}, \forall v \in \mathrm{L}^2(\Omega_\epsilon)$. So λ is away from the spectrum of A_{22}. And this particularly implies that $(A_{22} - \lambda)$ is invertible. Moreover one have

$$||(A_{22} - \lambda)^{-1}|| \leq C\epsilon^2$$

for some pure constant C.

$I - \tilde{\lambda}(A_{22} - \lambda)^{-1}$ is invertible for the following reasons. In the work [10], the authors proved

$$\lim_{\epsilon \to 0} \epsilon^{2\alpha_1} \left(\Lambda_j(\epsilon) - \frac{\pi^2}{M^2\epsilon^2} \right) = \mu_j \tag{5.4}$$

And we also proved that in Theorem 3.19 that

$$\lim_{\epsilon \to 0} \epsilon^{2\alpha_1} \left(\lambda_j - \frac{\pi^2}{M^2\epsilon^2} \right) = \mu_j \tag{5.5}$$

Combine (5.4) and (5.5) we have

$$\lim_{\epsilon \to 0} \epsilon^{2\alpha_1} \left(\Lambda_j - \lambda_j \right) = 0$$

namely

$$\lim_{\epsilon\to 0} \epsilon^{2\alpha_1}\tilde{\lambda} = 0 \tag{5.6}$$

Recall $\alpha_1 = \frac{2}{m+2} \leq \frac{1}{2}$ since m is even integer. So the fact $||(A_{22}-\lambda)^{-1}|| \leq C\epsilon^2$ combined with (5.6) will guarantee that $||\tilde{\lambda}(A_{22}-\lambda)^{-1}| < 1$ as $\epsilon \to 0$. And this shows $I - \tilde{\lambda}(A_{22}-\lambda)^{-1}$ is invertible and one even have the Neumann Expansion.

$$\left[I - \tilde{\lambda}(A_{22}-\lambda)^{-1}\right]^{-1} = \sum_{n=0}^{\infty} \tilde{\lambda}^n (A_{22}-\lambda)^{-n}$$

$\square$

Now we can state the equation that is satisfied by $\tilde{\lambda}$.

Lemma 5.1.2 (Equation for $\tilde{\lambda}$)**.** *Let ϕ be the normalized eigenfunctions of $A_{11}\phi = \lambda\phi$. Let $a_0 = -\frac{\langle u_1, A_{12}(A_{22}-\lambda)^{-1}A_{21}\phi\rangle}{\langle u_1,\phi\rangle}$, $a_n = -\frac{\langle u_1, A_{12}(A_{22}-\lambda)^{-n-1}A_{21}\phi\rangle}{\langle u_1,\phi\rangle}$ and we also define the function $g(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n$. Then*

$$\tilde{\lambda} = g(\tilde{\lambda}) \tag{5.7}$$

Proof. Notice

$$\tilde{\lambda} = \frac{\langle(\Delta-\lambda)u,\phi\rangle}{\langle u,\phi\rangle} = \frac{\langle(P\Delta P + P\Delta Q - \lambda)u, P\phi\rangle}{\langle u,\phi\rangle} = \frac{\langle u_2, \Delta\phi\rangle}{\langle u_1,\phi\rangle} = \frac{\langle A_{12}u_2,\phi\rangle}{\langle u_1,\phi\rangle}$$

Using (5.2), we have

$$-\tilde{\lambda}\langle u_1,\phi\rangle = \langle A_{12}(A_{22}-\lambda)^{-1}A_{21}u_1,\phi\rangle + \tilde{\lambda}\langle A_{12}(A_{22}-\lambda)^{-2}A_{21}u_1,\phi\rangle + \tilde{\lambda}^2\langle A_{12}(A_{22}-\lambda)^{-3}A_{21}u_1,\phi\rangle + \cdots$$

namely

$$-\tilde{\lambda}\langle u_1, \phi\rangle = \langle A_{12}(A_{22}-\lambda)^{-1}A_{21}u_1, \phi\rangle + \sum_{n=1}^{\infty}\tilde{\lambda}^n\langle A_{12}(A_{22}-\lambda)^{-n-1}A_{21}u_1, \phi\rangle$$

Then by self-adjointness we have

$$-\tilde{\lambda}\langle u_1, \phi\rangle = \langle u_1, A_{12}(A_{22}-\lambda)^{-1}A_{21}\phi\rangle + \sum_{n=1}^{\infty}\tilde{\lambda}^n\langle u_1, A_{12}(A_{22}-\lambda)^{-n-1}A_{21}\phi\rangle$$

Thus

$$-\tilde{\lambda} = \frac{\langle u_1, A_{12}(A_{22}-\lambda)^{-1}A_{21}\phi\rangle}{\langle u_1, \phi\rangle} + \sum_{n=1}^{\infty}\tilde{\lambda}^n\frac{\langle u_1, A_{12}(A_{22}-\lambda)^{-n-1}A_{21}\phi\rangle}{\langle u_1, \phi\rangle}$$

□

Remark 5.1.3. The lemma above tells us that $\tilde{\lambda}$ is a fixed point of the map $g(x)$. And in the next section we are going to show $g(x)$ is a contraction map, in particular we will show the series in defining the function $g(x)$ is convergent. And as a corallary we will have an iterative scheme for solving $\tilde{\lambda}$.

5.2 Solving the Equation

In this section we will show $g(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n$ is a contraction map. And we will also present the iterative scheme for solving the equation $g(\tilde{\lambda}) = \tilde{\lambda}$.

Lemma 5.2.1. $||u_1 - \phi||_{\mathbf{L}^2} = O(\epsilon^{3\alpha_1})$

Proof. See [10] page 5. □

Corollary 5.2.2. *As $\epsilon \to 0$, we have* $|\langle u_1, \phi\rangle| \geq \frac{1}{2}||u_1|| \cdot ||\phi||$.

Proof. Directly follows from Lemma 5.2.1. $\square$

Before we obtain the main estimates about the coefficients a_n we want to understand the operator A_{21} in more details.

Lemma 5.2.3. *Let* $f(x,y) = \chi(x)\sqrt{\frac{2}{\epsilon h(x)}}\sin(\frac{\pi y}{\epsilon h(x)}) \in \mathcal{L}_\epsilon$, *then for some pure constant C and D we have*

$$||A_{21}f(x,y)||^2 \leq C\int_{\mathbf{I}} \chi'^2 dx + D\int_{\mathbf{I}} \chi^2 dx$$

Proof. By direct computation and see Appendix B. $\square$

Now we prove a key Lemma which played an essential role in estimating the coefficients a_n.

Lemma 5.2.4. $\frac{||A_{21}u_1||\cdot||A_{21}\phi||}{||\langle u_1,\phi\rangle||} = O(\frac{1}{\epsilon^{2\alpha_1}})$.

Proof. Notice $\phi \in \mathcal{L}_\epsilon$, we can let $\phi = \chi(x)\sqrt{\frac{2}{\epsilon h(x)}}\sin(\frac{\pi y}{\epsilon h(x)})$ for some $\chi(x)$. Then $A_{11}\phi = \lambda\phi$ is equivalent to

$$[-\frac{d^2}{dx^2} + \frac{\pi^2}{\epsilon^2 h^2} + (\frac{\pi^2}{3} + \frac{1}{4})\frac{h'^2}{h^2}]\chi = \lambda\chi(x)$$

namely $-\chi'' = [\lambda - \frac{\pi^2}{\epsilon^2 h^2} - (\frac{\pi^2}{3} + \frac{1}{4})\frac{h'^2}{h^2}]\chi$. In particular, it implies that

$$\int \chi'^2 dx = \int [\lambda - \frac{\pi^2}{\epsilon^2 h^2} - (\frac{\pi^2}{3} + \frac{1}{4})\frac{h'^2}{h^2}]\chi^2 dx = O(\frac{\mu}{\epsilon^{2\alpha_1}})\int \chi^2 dx$$

since we have shown $\lambda - \frac{\pi^2}{\epsilon^2 M^2} \sim \frac{\mu}{\epsilon^{2\alpha_1}}$ in Theorem 4.3.8 (One can also refer to the two term asymptotics in [10]) where μ are eigenvalues of the operator on $L^2(\mathbf{R})$ given by $-\frac{d^2}{dx^2} + q(x)$ where $q(x) = 2\pi^2 M^{-3} c x^m$. So from Lemma 5.2.3, we have

$$\frac{||A_{21}\phi||}{||\phi||} = O(\frac{1}{\epsilon^{\alpha_1}})$$

Indeed we just showed that all the eigenfunctions of A_{11} would have similar estimate. In fact, let's assume $\{\xi_j\}_{j=0}^{\infty}$ be all the normalized eigenfunctions of A_{11} with corresponding eigenvalues $\{\lambda_j\}$. Then

$$\frac{||A_{21}\xi_j||}{||\xi_j||} = O(\frac{\sqrt{\mu_j}}{\epsilon^{\alpha_1}}) = O(\frac{\sqrt{\lambda_j}}{\epsilon^{\alpha_1}}) \tag{5.8}$$

But we know $\{\xi_j\}$ form a complete basis for $\mathcal{L}_\epsilon$. In particular, this allows us to show $\frac{||A_{21}u_1||}{||u_1||} = O(\frac{1}{\epsilon^{\alpha_1}})$. In fact, let $x_j = \langle u_1, \xi_j \rangle$. Then

$$u_1 = \sum_{j=0}^{\infty} x_j \xi_j, \quad ||u_1||^2 = \sum_j x_j^2 < \infty,$$

and we also have $||A_{11}^2 u_1||^2 = \sum_{j=0}^{\infty} (\lambda_j^2 x_j)^2 = \sum_{j=0}^{\infty} \lambda_j^4 x_j^2 < \infty$. Moreover,

$$
\begin{aligned}
||A_{21}u_1||^2 &= ||\sum_j x_j A_{21}\xi_j||^2 = \langle \sum_j x_j A_{21}\xi_j, \sum_k x_j A_{21}\xi_k \rangle \\
&= \sum_j x_j^2 ||A_{21}\xi_j||^2 + \sum_{j\neq k} x_j x_k \langle A_{21}\xi_j, A_{21}\xi_k\rangle \\
&\leq D\frac{1}{\epsilon^{2\alpha_1}} \left[\sum_j x_j^2 \mu_j + \sum_{j\neq k} |\sqrt{\mu_j} x_j| \cdot |\sqrt{\mu_k} x_k| \right] \\
&\leq D\frac{1}{\epsilon^{2\alpha_1}} \left(\sum_j |\sqrt{\mu_j} x_j| \right)^2 \\
&\leq D\frac{1}{\epsilon^{2\alpha_1}} \left(\sum_j |\mu_j^2 x_j \cdot \frac{1}{\mu_j \sqrt{\mu_j}}| \right)^2 \\
&\leq D\frac{1}{\epsilon^{2\alpha_1}} \left(\sum_j [\frac{|\mu_j^2 x_j|^p}{p} + \frac{1}{q}\frac{1}{\mu_j^{\frac{3q}{2}}}] \right)^2 = D\frac{1}{\epsilon^{2\alpha_1}} \left(\sum_j \frac{|\mu_j^2 x_j|^p}{p} + \sum_j \frac{1}{q}\frac{1}{\mu_j^{\frac{3q}{2}}} \right)^2
\end{aligned}
$$

where p, q are any positive conjugates, namely $\frac{1}{p} + \frac{1}{q} = 1$. And in the last inequality we are using classical inequality: $uv \leq \frac{u^p}{p} + \frac{v^q}{v}$ for $u, v > 0$. For our argument, let's fix $p = 4, q = \frac{4}{3}$. So we just showed

$$
||A_{21}u_1|| \leq D\frac{1}{\epsilon^{\alpha_1}} \left(\sum_j \frac{|\mu_j^2 x_j|^4}{4} + \sum_j \frac{3}{4}\frac{1}{\mu_j^2} \right) = D\frac{1}{\epsilon^{\alpha_1}} \left(\frac{1}{4}\sum_j |\mu_j^2 x_j|^4 + \frac{3}{4}\sum_j \frac{1}{\mu_j^2} \right)
$$

But notice $\mu_j \leq \epsilon^{2\alpha_1}\lambda_j$ and $\sum_j |\lambda_j^2 x_j|^4 < \infty$ as $\sum_{j=0}^{\infty} (\lambda_j^2 x_j)^2 = ||A_{11}^2 u_1||^2 < \infty$, hence

$$
\sum_j |\mu_j^2 x_j|^4 \leq O(\epsilon^{16\alpha_1}) \sum_j |\lambda_j^2 x_j|^4 < \infty
$$

Also notice $\sum_j \frac{1}{\lambda_j^2} < \infty$ since $\frac{1}{\lambda_j} \sim \frac{1}{\mu_j} \sim \frac{1}{j}$. In conclusion, for $\forall \gamma > 0, \exists N$, such that $\frac{1}{4}\sum_{j=N}^{\infty} |\mu_j^2 x_j|^4 < \gamma$. In particular, this implies $\frac{1}{4}\sum_{j=N}^{\infty} |\mu_j^2 x_j|^4 + \frac{3}{4}\sum_{j=0}^{\infty} \frac{1}{\mu_j^2} \leq \Gamma$ for some constant Γ which does not depend on ϵ. Thus

$$||A_{21}u_1|| \leq D\frac{1}{\epsilon^{\alpha_1}}\left(\Gamma + \sum_{j=0}^{N-1}\frac{1}{4}|\mu_j^2 x_j|^4\right)$$

Now let $\mu_* = \max\{\mu_0, \mu_1, \cdots, \mu_{N-1}\}$. Clearly μ_* does not depend on ϵ. Then

$$||A_{21}u_1|| \leq D\frac{1}{\epsilon^{\alpha_1}}\Gamma + \frac{D}{4}\frac{1}{\epsilon^{\alpha_1}}\mu_*^8\sum_{j=0}^{N-1}|x_j|^4 \leq O(\frac{1}{\epsilon^{\alpha_1}})||u_1||$$

In particualr, we have $\frac{||A_{21}u_1||}{||u_1||} = O(\frac{1}{\epsilon^{\alpha_1}})$.

To finish the proof, one simply need to notice in Corollary 5.2.2 we have

$$|\langle u_1, \phi\rangle| \geq \frac{1}{2}||u_1|| \cdot ||\phi||$$

And in summary, we just showed that $\frac{||A_{21}u_1||\cdot||A_{21}\phi||}{||\langle u_1,\phi\rangle||} = O(\frac{1}{\epsilon^{2\alpha_1}})$. □

Now we are ready to show that $g(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n$ is a contraction.

Theorem 5.2.5. *$g(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n$ is a contraction when ϵ is small enough.*

Proof. Recall $a_0 = -\frac{\langle u_1, A_{12}(A_{22}-\lambda)^{-1}A_{21}\phi\rangle}{\langle u_1,\phi\rangle}$, $a_n = -\frac{\langle u_1, A_{12}(A_{22}-\lambda)^{-n-1}A_{21}\phi\rangle}{\langle u_1,\phi\rangle}$. Hence

$$\begin{aligned}
|a_0| &= \frac{|\langle u_1, A_{12}(A_{22}-\lambda)^{-1}A_{21}\phi\rangle|}{|\langle u_1, \phi\rangle|} = \frac{|\langle A_{21}u_1, (A_{22}-\lambda)^{-1}A_{21}\phi\rangle|}{|\langle u_1, \phi\rangle|} \\
&\leq \frac{||A_{21}u_1|| \cdot ||A_{21}\phi||}{||\langle u_1, \phi\rangle||}||(A_{22}-\lambda)^{-1}|| \\
|a_n| &= \frac{|\langle u_1, A_{12}(A_{22}-\lambda)^{-n-1}A_{21}\phi\rangle|}{|\langle u_1, \phi\rangle|} = \frac{|\langle A_{21}u_1, (A_{22}-\lambda)^{-n-1}A_{21}\phi\rangle|}{|\langle u_1, \phi\rangle|} \\
&\leq \frac{||A_{21}u_1|| \cdot ||A_{21}\phi||}{||\langle u_1, \phi\rangle||}||(A_{22}-\lambda)^{-n-1}||
\end{aligned}$$

By **Lemma 5.1.1**, we know for some constant C, which does not depend on ϵ, we have

$$||(A_{22}-\lambda)^{-1}|| \leq C\epsilon^2, \quad ||(A_{22}-\lambda)^{-n-1}|| \leq (C\epsilon^2)^{(n+1)}$$

Combining with **Lemma 5.2.4**, we have

$$|a_0| \leq C\frac{||A_{21}u_1||\cdot||A_{21}\phi||}{||\langle u_1,\phi\rangle||}\epsilon^2 \leq C\epsilon^{2-2\alpha_1} \tag{5.9}$$

$$|a_n| \leq \frac{||A_{21}u_1||\cdot||A_{21}\phi||}{||\langle u_1,\phi\rangle||}(C\epsilon^2)^{(n+1)} \leq (C\epsilon^2)^{n+1}\epsilon^{-2\alpha_1} \tag{5.10}$$

where one might redefine the constant C as needed. With the estimates of the coefficients a_0, a_n of function $g(x)$, it is easy to prove the following fact which will also show $g(x)$ is a contraction automatically.

$$g'(x) = \textstyle\sum_{n=1}^{\infty} na_n x^{n-1} \text{ satisfy } |g'(x)| < \frac{1}{2} \text{ for all } |x| \leq \frac{C}{\epsilon^{2\alpha_1}}$$

Indeed we have

$$\begin{aligned}|g'(x)| &\leq \frac{||A_{21}u_1||\cdot||A_{21}\phi||}{||\langle u_1,\phi\rangle||}\sum_{n=1}^{\infty} n(C\epsilon^2)^{n+1}|x|^{n-1} \leq \frac{||A_{21}u_1||\cdot||A_{21}\phi||}{||\langle u_1,\phi\rangle||}\epsilon^{2+2\alpha_1}\sum_{n=1}^{\infty} n(C\epsilon^{1-\alpha_1})^{2n}\\ &\leq \frac{||A_{21}u_1||\cdot||A_{21}\phi||}{||\langle u_1,\phi\rangle||}\frac{C^2\epsilon^{2-2\alpha_1}}{(1-C^2\epsilon^{2-2\alpha_1})^2}\cdot\epsilon^{2+2\alpha_1}\end{aligned}$$

Now using **Lemma 5.2.4** we have

$$|g'(x)| \leq O(\frac{1}{\epsilon^{2\alpha_1}})\frac{C^2\epsilon^{2-2\alpha_1}}{(1-C^2\epsilon^{2-2\alpha_1})^2}\cdot\epsilon^{2+2\alpha_1}$$

So $|g'(x)| \to 0$ as $\epsilon \to 0$.

$\square$

Corollary 5.2.6. *Let λ be eigenvalues of A_{11} with normalized eigenfunction ϕ. We also let $\tilde{\lambda} = \Lambda_\epsilon - \lambda$. Then $\tilde{\lambda}_n \to \tilde{\lambda}$ as $n \to \infty$, where*

$$\tilde{\lambda}_0 = a_0$$

$$\tilde{\lambda}_{n+1} = g(\tilde{\lambda}_n)$$

and $g(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n$ *with* $a_0 = -\frac{\langle u_1, A_{12}(A_{22}-\lambda)^{-1} A_{21}\phi\rangle}{\langle u_1, \phi\rangle}, a_n = -\frac{\langle u_1, A_{12}(A_{22}-\lambda)^{-n-1} A_{21}\phi\rangle}{\langle u_1, \phi\rangle}$.

Proof. Recall Lemma 5.1.2 where we show that $\tilde{\lambda} = g(\tilde{\lambda})$. So the corollary follows directly from Theorem 5.2.5 where we showed that $g(x)$ is a contraction and the fact that $\lim_{\epsilon \to 0} \epsilon^{2\alpha_1} \tilde{\lambda} = 0$ from Lemma 5.1.1. $\square$

Theorem 5.2.7. $|\tilde{\lambda}| = O(\epsilon^{2-2\alpha_1})$. *In particular* $\lim_{\epsilon\to 0}\tilde{\lambda} = 0$.

Proof. Notice form Lemma 5.1 we know $\lim_{\epsilon\to 0}\epsilon^{2\alpha_1}\tilde{\lambda} = 0$, in particular we have

$$|\tilde{\lambda}| \le \frac{C}{\epsilon^{2\alpha_1}} \tag{5.11}$$

Notice also $\tilde{\lambda} = g(\tilde{\lambda})$. Thus

$$\begin{aligned}
|\tilde{\lambda}| &\le |g(\tilde{\lambda})| \\
&\le |a_0| + \sum_{n=1}^{\infty} |a_n| \cdot |\tilde{\lambda}|^n \\
&\le C\epsilon^{2-2\alpha_1} + \sum_{n=1}^{\infty} (C\epsilon^2)^{n+1}\epsilon^{-2\alpha_1}|\tilde{\lambda}|^n \\
&\le C\epsilon^{2-2\alpha_1} + \sum_{n=1}^{\infty} (C\epsilon^2)^{n+1}\epsilon^{-2\alpha_1}\left(\frac{C}{\epsilon^{2\alpha_1}}\right)^n \\
&\le C\epsilon^{2-2\alpha_1}
\end{aligned}$$

where we use estimates (5.9) and (5.10) in the third inequality and we use (5.11) in the fourth inequality. $\square$

Remark 5.2.8. In the case when one has detailed asymptotics of the coefficients a_i's following Corollary 5.2.6 one will obtain detailed asymptotics of $\tilde{\lambda}$. And this would be studied later in some other places.

CHAPTER 6

Future Directions

This chapter is largely speculative. We summarize the problems left open in this dissertation and provide some hints to the solutions. There are some further research that can be continued from this project. More precisely the asymptotic expansion of the eigenvalues that we get in Theorem B is not convergent as one easily see. But one can ask for a proper summation method to make sense of this divergent series. For instance one can study the **Borel Summability** here. The main things involved here is **Watson's Theorem** as introduced in [21].

Definition 6.0.1. A function $E(\beta)$ analytic in a sectorial region $\{\beta | 0 < |\beta| < B, |\arg\ \beta| \le \frac{1}{2}\pi + \epsilon\}$ obeys a strong asymptotic condition and has $\sum_{n=0}^{\infty} a_n \beta^n$ as **strong asymptotic series** if there are C and σ so that

$$|E(\beta) - \sum_{n=0}^{N} a_n \beta^n| \le C\sigma^{N+1}(N+1)!|\beta|^{N+1} \tag{6.1}$$

for all N and all β in the sector.

Theorem 6.0.2. *(**Watson's Theorem:**) Suppose $E(\beta)$ has $\sum_{n=0}^{\infty} a_n \beta^n$ as strong asymptotic series in $\{\beta | 0 < |\beta| < B, |arg\ \beta| \le \frac{1}{2}\pi + \epsilon\}$. Then the function $g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is analytic in some circle about $z = 0$. And*

1. *g has an analytic continuation to the region* $\{z : |arg\ z| < \epsilon\}$

2. *If* $|\beta| < B$ *and* $|arg\ \beta| < \epsilon$*, then* $\int_0^\infty |g(x\beta)|e^{-x}dx < \infty$

3. *If* $|\beta| < B$ *and* $|arg\ \beta| < \epsilon$*, then* $E(\beta) = \int_0^\infty g(x\beta)e^{-x}dx$

Notice the key condition for the Borel Summation Method to work is the estimate (6.1). And for our particular case, if we can find estimate "$|q_n| \leq C\sigma^{N+1}(N+1)!$" then that is a strong hint to suggest that our asymptotic series in Theorem B is Borel Summable. To obtain Borel summability one also of course has to obtain analyticity of the eigenvalues for $\epsilon^{2\alpha_1}\left(A_{11} - \frac{\pi^2}{M^2\epsilon^2}\right)$ in a cone domain which is having angle larger than π. We now will present some partial results along these lines.

Lemma 6.0.3. *For $m = 2$, we have $|q_n| \le Dn!\sigma^n, \forall n$ where D, σ are some constants.*

Proof. Notice $H_0 = -\frac{d^2}{dy^2} + 2a_0a_1y^2$ is the most familiar harmonic oscillator.Moreover by making changing variable $y \to \omega y$ where $\omega = \sqrt[4]{\frac{1}{2a_0a_1}}$, one changes

$$H_0 \to \sqrt{2a_0a_1}\left(-\frac{d^2}{dy^2} + y^2\right).$$

So without loss of generality one may assume

$$H_0 = -\frac{d^2}{dy^2} + y^2, \quad H_n = (n+2)a_0a_1^{n+1}\omega^{2n+2}y^{2n+2} + (n-1)aa_0a_1^{n-2}\omega^{2n-2}y^{2n-2}.$$

Recall that $H_0 = A^\dagger A + \frac{1}{2}$, where $A^\dagger = \frac{1}{\sqrt{2}}(-\frac{d}{dy} + y), A = \frac{1}{\sqrt{2}}(\frac{d}{dy} + y)$. Moreover the eigenstates $\{\Omega_j\}_{j=0}^{\infty}$ of H_0 form a complete basis with

$$H_0\Omega_j = (2j+1)\Omega_j, j = 0, 1, \cdots$$

$$A\Omega_j = \sqrt{j}\Omega_{j-1}, j \ge 1$$

$$A^\dagger\Omega_j = \sqrt{j+1}\Omega_{j+1}, j = 0, 1, \cdots$$

Clearly one also has $y^2 = \frac{1}{2}(A^\dagger + A)^2$. Now let's focus on

$$\begin{aligned} &\mathbf{Tr}\int_\Gamma (-1)^k\lambda(H_0-\lambda)^{-1}H_{j_1}(H_0-\lambda)^{-1}\cdots H_{j_k}(H_0-\lambda)^{-1}d\lambda \\ =&\sum_{s=0}^{\infty}\langle\int_\Gamma (-1)^k\lambda(H_0-\lambda)^{-1}H_{j_1}(H_0-\lambda)^{-1}\cdots H_{j_k}(H_0-\lambda)^{-1}d\lambda\Omega_s, \Omega_s\rangle \end{aligned}$$

Without loss of generality assume Γ is encloseing only $\mu_T = 2T+1$. Notice for any j,

$$
\begin{aligned}
H_j &= (j+2)a_0 a_1^{j+1}\omega^{2j+2}y^{2j+2} + (j-1)aa_0a_1^{j-2}\omega^{2j-2}y^{2j-2} \\
&= (j+2)a_0 a_1^{j+1}\omega^{2j+2}\left[\frac{1}{2}(A^\dagger + A)^2\right]^{j+1} + (j-1)aa_0a_1^{j-2}\omega^{2j-2}\left[\frac{1}{2}(A^\dagger + A)^2\right]^{j-1}
\end{aligned}
$$

which shows that H_j can raise or lower the engergy level at most by $2(j+1)$. Also notice we are taking $\int_\Gamma d\lambda$ in computing the trace so whatever Ω_s it starts from, through the operation of $(H_0-\lambda)^{-1}H_{j_1}(H_0-\lambda)^{-1}\cdots H_{j_k}(H_0-\lambda)^{-1}$, it has to lands on Ω_T at least once. Moreover because we are taking $\langle\cdot\Omega_s,\Omega_s\rangle$, so through the operation $(H_0-\lambda)^{-1}H_{j_1}(H_0-\lambda)^{-1}\cdots H_{j_k}(H_0-\lambda)^{-1}$ on Ω_s, it has to ends on Ω_s. Hence

$$
\begin{aligned}
&\mathbf{Tr}\int_\Gamma (-1)^k\lambda(H_0-\lambda)^{-1}H_{j_1}(H_0-\lambda)^{-1}\cdots H_{j_k}(H_0-\lambda)^{-1}d\lambda \\
=&\sum_{s=0}^{\infty}\langle\int_\Gamma (-1)^k\lambda(H_0-\lambda)^{-1}H_{j_1}(H_0-\lambda)^{-1}\cdots H_{j_k}(H_0-\lambda)^{-1}d\lambda\Omega_s,\Omega_s\rangle \\
=&\sum_{|s-T|\le\frac{n}{2}}\langle\int_\Gamma (-1)^k\lambda(H_0-\lambda)^{-1}H_{j_1}(H_0-\lambda)^{-1}\cdots H_{j_k}(H_0-\lambda)^{-1}d\lambda\Omega_s,\Omega_s\rangle \\
\le&D(T+1)(T+2)\cdots(T+\frac{n}{2}) \\
\le&Dn!\sigma^n
\end{aligned}
$$

Thus

$$
\begin{aligned}
|q_n| &= \sum_{k=1}^{n}\frac{1}{2\pi i}\mathbf{Tr}\int_\Gamma\left(\sum_{j_1+j_2+\cdots+j_k=n,j_i\in\mathbf{Z}^+}(-1)^k\lambda(H_0-\lambda)^{-1}H_{j_1}(H_0-\lambda)^{-1}H_{j_2}(H_0-\lambda)^{-1}\cdots H_{j_k}(H_0-\lambda)^{-1}\right)d\lambda \\
&\le\sum_{k=1}^{n}\sum_{j_1+j_2+\cdots+j_k=n,j_i\in\mathbf{Z}^+}Dn!\sigma^n \\
&\le Dn!\sigma^n
\end{aligned}
$$

for the last inequlatiy we are using asymptotics for partition function

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$$

and modifying the constants D, σ if needed. □

Now let's look at the case where $m \geq 4$, m an even integer. Without loss of generality one may assume $H_0 = -\frac{d^2}{dy^2} + y^{2m}$. Motivated from the case for harmonic oscillator, we let $A = \frac{d}{dy} + y^m, A^\dagger = -\frac{d}{dy} + y^m$. Then one has the following

$$H_0 = A^\dagger A + R$$

$$H_0 = AA^\dagger - R$$

$$[A, A^\dagger] = 2R$$

where $R = my^{m-1}$. It is easy to observe that we have

$$\begin{aligned}
y^m &= \frac{1}{2}(A + A^\dagger) \\
y^{m-1} &= \frac{1}{m}[A, y^m] = \frac{1}{2m}[A, A^\dagger] = \frac{1}{m}(H_0 - A^\dagger A) \\
y^{m-2} &= \frac{1}{m-1}[A, y^{m-1}] \\
y^{m-i} &= \frac{1}{m-i+1}[A, y^{m-i+1}] = \frac{1}{m-i+1}[y^{m-i+1}, A^\dagger] \\
y &= \frac{1}{2}[A, y^2]
\end{aligned}$$

Based on these observations one might tend to make a similar argument as in the Lemma above and try to obtain similar bounds for $|q_n|$ in this case. But we are missing the ingredient that $A = \frac{d}{dy} + y^m, A^\dagger = -\frac{d}{dy} + y^m$ as defined play the role of "lowering" and "raising" operator compared with the harmonic oscillator case where $m = 2$. Along this direction the following results are worth noticing.

Lemma 6.0.4. *Let $S \in CL^r_{\alpha,p}$ be an elliptic pseudodifferential operator with positive principal symbol and $m > 0$. Then there exist operators $a^\pm \in CL^{r/2}_{\alpha,q}$ (creation and annilation operators), such that*

- $ord(a^+ - a^{-*}) < r$;
- $S = a^+a^- + R_1$;
- $[a^-, a^+] = f(S) + R_2$;
- $ind\ a^- = 1$

where R_1 and R_2 are negligible operators and

$$f(\lambda) \sim \sum_{j=0}^{\infty} c_j \lambda^{\sigma_j}, \lambda \to \infty,$$

with $\sigma_j = \frac{r-1-j/q}{r}$. Moreover if the operator S is selfadjoint and if the spectrum of this operator is asymptotically simple (meaning that the operator S has at most a finite number of multiple eigenvalues), then one can choose $a^\pm$ in such a way that $R_2 = 0$.

Proof. See [9]. $\square$

Lemma 6.0.5. *$H_0 = -\frac{d^2}{dy^2} + 2a_0a_1y^{2m}$ can be approximately factored as products of creation and annihilation operators. More precisely, there exist operators a^+, a (creation and annihilation operators) such that*

1. $H_0 = a^+a + R$

2. $[a, a^+] = \beta_0 H_0^{\frac{m-1}{2m}}$ *where* $\beta_0 = \frac{2\pi \cdot \frac{2m}{m+1}}{\int_{\xi^2 + x^{2 \cdot \frac{2m}{m+1}} \leq 1} dx d\xi}$

Proof. Follow as a corollary [9]. $\square$

CHAPTER 7

Appendix of Proof of Lemmas

7.1 Proof of Lemma 4.3.1

Proof. Taylor Expansion. More precisely, notice that $h(x) = M - cx^m$ and $\frac{\pi^2}{\epsilon^2 h^2} = \frac{\pi^2}{M^2\epsilon^2}\left[\sum_{n=1}^{\infty} n \left(\frac{cx^m}{M}\right)^{n-1}\right]$, so we have

$$\begin{aligned} A_\epsilon &= -\frac{d^2}{dx^2} + \frac{\pi^2}{\epsilon^2 h^2} - \frac{\pi^2}{\epsilon^2 M^2} + (\frac{\pi^2}{3} + \frac{1}{4})\frac{h'^2}{h^2} \\ &= -\frac{d^2}{dx^2} + \frac{\pi^2}{M^2\epsilon^2}\left[\sum_{n=2}^{\infty} n \left(\frac{cx^m}{M}\right)^{n-1}\right] + (\frac{\pi^2}{3} + \frac{1}{4})\frac{m^2c^2x^{2(m-1)} + c'^2x^{2m} + 2mcc'x^{2m-1}}{M^2}\left[\sum_{n=1}^{\infty} n \left(\frac{cx^m}{M}\right)^{n-1}\right] \end{aligned}$$

Buy introducing $x = \epsilon^{\alpha_1} y$ where $\alpha_1 = \frac{2}{m+2}, y \in \mathbf{I}_\epsilon = [-\frac{l_1}{\epsilon^{\alpha_1}}, \frac{l_2}{\epsilon^{\alpha_1}}]$, we see that

$$
\begin{aligned}
\epsilon^{2\alpha_1} A_\epsilon &= -\frac{d^2}{dy^2} + 2a_0 a_1 y^m + \sum_{n=1}^{\infty} \left[(n+2)a_0 a_1^{n+1} y^{nm+m} + (n-1) a a_0 a_1^{n-2} y^{nm-2}\right] \epsilon^{n\alpha} \\
&+ \sum_{n=1}^{\infty} (n-1) b_1 a_1^{n-2} y^{nm-1} \epsilon^{n\alpha+\alpha_1} + \sum_{n=1}^{\infty} (n-1) b_2 a_1^{n-2} y^{nm} \epsilon^{n\alpha+2\alpha_1} \\
&= -\frac{d^2}{dy^2} + 2a_0 \left(\sum_{n=0}^{\infty} \frac{c_n}{M} y^n \epsilon^{n\alpha_1}\right) y^m + \sum_{n=1}^{\infty} \epsilon^{nm\alpha_1} \left[(n+2)a_0 \left(\sum_{k=0}^{\infty} \alpha_{k,n} y^k \epsilon^{k\alpha_1}\right) y^{nm+m}\right] \\
&+ \sum_{n=1}^{\infty} \epsilon^{nm\alpha_1} \left[(\frac{\pi^2}{3} + \frac{1}{4}) \frac{m^2}{\pi^2} (n-1) a_0 \left(\sum_{n=0}^{\infty} d_n y^n \epsilon^{n\alpha_1}\right) \left(\sum_{k=0}^{\infty} \beta_{k,n} y^k \epsilon^{k\alpha_1}\right)\right] \\
&+ \sum_{n=1}^{\infty} \epsilon^{nm\alpha_1+\alpha_1} \left[\frac{2m}{M^2} (\frac{\pi^2}{3} + \frac{1}{4})(n-1) \left(\sum_{n=0}^{\infty} f_n y^n \epsilon^{n\alpha_1}\right) \left(\sum_{k=0}^{\infty} \beta_{k,n} y^k \epsilon^{k\alpha_1}\right)\right] \\
&+ \sum_{n=1}^{\infty} \epsilon^{nm\alpha_1+2\alpha_1} \left[(\frac{\pi^2}{3} + \frac{1}{4})(n-1) \frac{1}{M^2} \left(\sum_{n=0}^{\infty} g_n y^n \epsilon^{n\alpha_1}\right) \left(\sum_{k=0}^{\infty} \beta_{k,n} y^k \epsilon^{k\alpha_1}\right)\right] \\
&= -\frac{d^2}{dy^2} + 2a_0 \left(\sum_{n=0}^{\infty} \frac{c_n}{M} y^n \epsilon^{n\alpha_1}\right) y^m + \sum_{n=1}^{\infty} \epsilon^{nm\alpha_1} \left[(n+2)a_0 \left(\sum_{k=0}^{\infty} \alpha_{k,n} y^k \epsilon^{k\alpha_1}\right) y^{nm+m}\right] \\
&+ \sum_{n=1}^{\infty} \epsilon^{nm\alpha_1} \left(\sum_{k=0}^{\infty} \beta_{k,n} y^k \epsilon^{k\alpha_1}\right) \left[A_1 \left(\sum_{n=0}^{\infty} d_n y^n \epsilon^{n\alpha_1}\right) + A_2 \epsilon^{\alpha_1} \left(\sum_{n=0}^{\infty} f_n y^n \epsilon^{n\alpha_1}\right) + A_3 \epsilon^{2\alpha_1} \left(\sum_{n=0}^{\infty} g_n y^n \epsilon^{n\alpha_1}\right)\right] \\
&= -\frac{d^2}{dy^2} + 2a_0 \left(\sum_{n=0}^{\infty} \frac{c_n}{M} y^n \epsilon^{n\alpha_1}\right) y^m + \sum_{n=1}^{\infty} \epsilon^{nm\alpha_1} \left[(n+2)a_0 \left(\sum_{k=0}^{\infty} \alpha_{k,n} y^k \epsilon^{k\alpha_1}\right) y^{nm+m}\right] \\
&+ \sum_{n=1}^{\infty} \epsilon^{nm\alpha_1} \left(\sum_{k=0}^{\infty} \beta_{k,n} y^k \epsilon^{k\alpha_1}\right) \left[\sum_{k=0}^{\infty} \left(A_1 d_k y^k + A_2 f_{k-1} y^{k-1} + A_3 g_{k-2} y^{k-2}\right) \epsilon^{k\alpha_1}\right] \\
&= -\frac{d^2}{dy^2} + 2a_0 \left(\sum_{n=0}^{\infty} \frac{c_n}{M} y^n \epsilon^{n\alpha_1}\right) y^m + \sum_{n=1}^{\infty} \epsilon^{nm\alpha_1} \left[(n+2)a_0 \left(\sum_{k=0}^{\infty} \alpha_{k,n} y^k \epsilon^{k\alpha_1}\right) y^{nm+m}\right] \\
&+ \sum_{n=1}^{\infty} \epsilon^{nm\alpha_1} \left(\sum_{k=0}^{\infty} \beta_{k,n} y^k \epsilon^{k\alpha_1}\right) \left[\sum_{k=0}^{\infty} \gamma_k \epsilon^{k\alpha_1}\right] \\
&= -\frac{d^2}{dy^2} + 2a_0 \left(\sum_{n=0}^{\infty} \frac{c_n}{M} y^n \epsilon^{n\alpha_1}\right) y^m + \sum_{n=1}^{\infty} \epsilon^{nm\alpha_1} \left[(n+2)a_0 \left(\sum_{k=0}^{\infty} \alpha_{k,n} y^k \epsilon^{k\alpha_1}\right) y^{nm+m}\right] \\
&+ \sum_{n=1}^{\infty} \epsilon^{nm\alpha_1} \sum_{k=0}^{\infty} \left(\sum_{i+j=k, i,j \in \mathbf{N}} \beta_{i,n} y^i \gamma_j\right) \epsilon^{k\alpha_1} \\
&= -\frac{d^2}{dy^2} + 2a_0 \left(\sum_{n=0}^{\infty} \frac{c_n}{M} y^n \epsilon^{n\alpha_1}\right) y^m \\
&+ \sum_{n=1}^{\infty} \epsilon^{nm\alpha_1} \sum_{k=0}^{\infty} \left(\left[(n+2)a_0 \alpha_{k,n} y^{k+nm+m} + \sum_{i+j=k, i,j \in \mathbf{N}} \beta_{i,n} y^i \gamma_j\right)\right] \epsilon^{k\alpha_1} \\
&= -\frac{d^2}{dy^2} + \left(\sum_{n=0}^{\infty} \frac{2a_0 c_n}{M} y^{n+m} \epsilon^{n\alpha_1}\right) \\
&+ \sum_{n=m}^{\infty} \sum_{k+sm=n, k \in \mathbf{N}, s \geq 1} \left[(s+2)a_0 \alpha_{k,s} y^{n+m} + \sum_{i+j=k, i,j \in \mathbf{N}} \beta_{i,s} y^i \gamma_j\right] \epsilon^{n\alpha_1} \\
&= -\frac{d^2}{dy^2} + \frac{2a_0 c_0}{M} y^m + \sum_{n=1}^{\infty} H_n \epsilon^{n\alpha_1}
\end{aligned}
$$

with $b_1 = (\frac{\pi^2}{3} + \frac{1}{4}) \frac{2mc(\epsilon^{\alpha_1} y) c'(\epsilon^{\alpha_1} y)}{M^2}, b_2 = (\frac{\pi^2}{3} + \frac{1}{4}) \frac{c'^2(\epsilon^{\alpha_1} y)}{M^2}$, $\alpha = m\alpha_1 = \frac{2m}{m+2}, a_0 =$

$\frac{\pi^2}{M^2}$, $a_1 = \frac{c(\epsilon^{\alpha_1}y)}{M} a = (\frac{\pi^2}{3} + \frac{1}{4})\frac{m^2c^2(\epsilon^{\alpha_1}y)}{\pi^2}$ and

$$H_n = \frac{2a_0c_n}{M}y^{n+m} + \sum_{k+sm=n,k\in\mathbf{N},s\geq 1}\left[(s+2)a_0\alpha_{k,s}y^{n+m} + \sum_{i+j=k,i,j\in\mathbf{N}}\beta_{i,s}y^i\gamma_j\right]$$

where $\gamma_j = A_1d_jy^j + A_2f_{j-1}y^{j-1} + A_3g_{j-2}y^{j-2}$. □

7.2 Proof of Lemma 5.2.3

Lemma 7.2.1. ***(Explicit Computation of*** A_{21}***)***

Let $f(x,y) = \chi(x)\sqrt{\frac{2}{\epsilon h(x)}}\sin(\frac{\pi y}{\epsilon h(x)}) \in \mathcal{L}_\epsilon$, *we also let* $g(x,y) = \sqrt{\frac{2}{\epsilon h(x)}}\sin(\frac{\pi y}{\epsilon h(x)})$, *then*

$$\begin{aligned}||A_{21}f(x,y)||^2 &= \int_{\mathbf{I}} 4\left(\frac{\pi^2}{3}+\frac{1}{4}\right)\frac{h'^2}{h^2}\chi'^2 dx + \int_{\mathbf{I}} 4\chi\chi'\left[\frac{h'}{h}\frac{h''}{h}\left(\frac{\pi^2}{3}+\frac{1}{4}\right) + \left(\frac{h'}{h}\right)^3\left(-\frac{1}{4}-\frac{4}{3}\pi^2\right)\right]dx \\ &+ \int_{\mathbf{I}} \chi^2\left[\left(\frac{1}{2}+\frac{53\pi^2}{12}+\frac{\pi^4}{5}\right)\left(\frac{h'}{h}\right)^4 - \left(\frac{1}{2}+\frac{8\pi^2}{3}\right)\left(\frac{h'}{h}\right)^2\frac{h''}{h}\right]dx \\ &+ \int_{\mathbf{I}} \chi^2\left[\left(\frac{\pi^2}{3}+\frac{1}{4}\right)\left(\frac{h''}{h}\right)^2 - \left(\frac{\pi^2}{3}+\frac{1}{4}\right)^2\left(\frac{h'}{h}\right)^4 + \left(\frac{1}{16}+\frac{1}{12}\pi^2\right)\left(\frac{h'}{h}\right)^3\right]dx\end{aligned}$$

Proof. By computation we have

1. $\Delta f = \chi''g + \chi g'' + 2\chi'g' + \chi\left(\frac{\pi}{\epsilon h}\right)^2 g$

2.

$$\begin{aligned}P\Delta f &= \left[\int_0^{\epsilon h}\left(\chi''g + \chi g'' + 2\chi'g' + \chi\left(\frac{\pi}{\epsilon h}\right)^2 g\right) g dy\right] g \\ &= \left[\chi'' + \chi\left(\frac{\pi}{\epsilon h}\right)^2 - \chi\left(\frac{\pi^2}{3}+\frac{1}{4}\right)\frac{h'^2}{h^2}\right] g\end{aligned}$$

3.

$$
\begin{aligned}
||P\Delta f||^2 &= \int_{\Omega_\epsilon} \left[\chi'' + \chi\left(\frac{\pi}{\epsilon h}\right)^2 - \chi\left(\frac{\pi^2}{3}+\frac{1}{4}\right)\frac{h'^2}{h^2}\right]^2 g^2 dxdy \\
&= \int_{\mathbf{I}} \left[\chi'' + \chi\left(\frac{\pi}{\epsilon h}\right)^2 - \chi\left(\frac{\pi^2}{3}+\frac{1}{4}\right)\frac{h'^2}{h^2}\right]^2 dx
\end{aligned}
$$

4. we also have

$$
\begin{aligned}
||\Delta f||^2 &= \langle \chi'' g + \chi g'' + 2\chi' g' + \chi\left(\frac{\pi}{\epsilon h}\right)^2 g, \chi'' g + \chi g'' + 2\chi' g' + \chi\left(\frac{\pi}{\epsilon h}\right)^2 g\rangle \\
&= \int_{\mathbf{I}} \left[\chi'' + \chi\left(\frac{\pi}{\epsilon h}\right)^2\right]^2 dx - 2\int_{\mathbf{I}}\left(\frac{\pi^2}{3}+\frac{1}{4}\right)\frac{h'^2}{h^2}\chi\left[\chi''+\chi\left(\frac{\pi}{\epsilon h}\right)^2\right]dx + \int_{\mathbf{I}} 4\chi'^2\left(\int_0^{\epsilon h} g'^2 dy\right)dx \\
&\quad + \int_{\mathbf{I}} 4\chi\chi'\left(\int_0^{\epsilon h} g' g'' dy\right)dx + \int_{\mathbf{I}}\chi^2\left(\int_0^{\epsilon h} g''^2 dy\right)dx \\
&= \int_{\mathbf{I}} \left[\chi'' + \chi\left(\frac{\pi}{\epsilon h}\right)^2\right]^2 dx - 2\int_{\mathbf{I}}\left(\frac{\pi^2}{3}+\frac{1}{4}\right)\frac{h'^2}{h^2}\chi\left[\chi''+\chi\left(\frac{\pi}{\epsilon h}\right)^2\right]dx + \int_{\mathbf{I}} 4\left(\frac{\pi^2}{3}+\frac{1}{4}\right)\frac{h'^2}{h^2}\chi'^2 dx \\
&\quad + \int_{\mathbf{I}} 4\chi\chi'\left[\frac{h'}{h}\frac{h''}{h}\left(\frac{\pi^2}{3}+\frac{1}{4}\right) + \left(\frac{h'}{h}\right)^3\left(-\frac{1}{4}-\frac{4}{3}\pi^2\right)\right]dx \\
&\quad + \int_{\mathbf{I}}\chi^2\left[\left(\frac{1}{2}+\frac{53\pi^2}{12}+\frac{\pi^4}{5}\right)\left(\frac{h'}{h}\right)^4 - \left(\frac{1}{2}+\frac{8\pi^2}{3}\right)\left(\frac{h'}{h}\right)^2\frac{h''}{h} + \left(\frac{\pi^2}{3}+\frac{1}{4}\right)\left(\frac{h''}{h}\right)^2\right]dx
\end{aligned}
$$

Thus

$$
\begin{aligned}
||A_{21}f(x,y)||^2 &= ||Q\Delta f(x,y)||^2 = ||\Delta f||^2 - ||P\Delta f||^2 \\
&= \int_{\mathbf{I}} 4\left(\frac{\pi^2}{3}+\frac{1}{4}\right)\frac{h'^2}{h^2}\chi'^2 dx + \int_{\mathbf{I}} 4\chi\chi'\left[\frac{h'}{h}\frac{h''}{h}\left(\frac{\pi^2}{3}+\frac{1}{4}\right) + \left(\frac{h'}{h}\right)^3\left(-\frac{1}{4}-\frac{4}{3}\pi^2\right)\right]dx \\
&\quad + \int_{\mathbf{I}}\chi^2\left[\left(\frac{1}{2}+\frac{53\pi^2}{12}+\frac{\pi^4}{5}\right)\left(\frac{h'}{h}\right)^4 - \left(\frac{1}{2}+\frac{8\pi^2}{3}\right)\left(\frac{h'}{h}\right)^2\frac{h''}{h}\right]dx \\
&\quad + \int_{\mathbf{I}}\chi^2\left[\left(\frac{\pi^2}{3}+\frac{1}{4}\right)\left(\frac{h''}{h}\right)^2 - \left(\frac{\pi^2}{3}+\frac{1}{4}\right)^2\left(\frac{h'}{h}\right)^4 + \left(\frac{1}{16}+\frac{1}{12}\pi^2\right)\left(\frac{h'}{h}\right)^3\right]dx
\end{aligned}
$$

where

$$g' = \frac{\partial}{\partial x} g(x,y) = -\frac{1}{2} \cdot \frac{h'}{h} g - \frac{h'}{h} \frac{\pi y}{\epsilon h} \sqrt{\frac{2}{\epsilon h(x)}} \cos(\frac{\pi y}{\epsilon h(x)})$$

$$\begin{aligned} g'' &= \frac{\partial^2}{\partial x^2} g(x,y) \\ &= -\frac{1}{2} \left(\frac{h''}{h} - \frac{h'^2}{h^2} \right) g - \frac{1}{2} \frac{h'}{h} g' - \frac{\pi y}{\epsilon h} \left(\frac{h''}{h} - \frac{5}{2} \frac{h'^2}{h^2} \right) \sqrt{\frac{2}{\epsilon h(x)}} \cos(\frac{\pi y}{\epsilon h(x)}) \\ &\quad - \frac{h'^2}{h^2} \left(\frac{\pi y}{\epsilon h} \right)^2 \sqrt{\frac{2}{\epsilon h(x)}} \sin(\frac{\pi y}{\epsilon h(x)}) \end{aligned}$$

and

$$\int_0^{\epsilon h} g^2 dy = 1$$

$$\int_0^{\epsilon h} g'^2 dy = \frac{h'^2}{h^2} \left(\frac{1}{4} + \frac{1}{3}\pi^2 \right)$$

$$\int_0^{\epsilon h} g''^2 dy = \left(\frac{1}{2} + \frac{53\pi^2}{12} + \frac{\pi^4}{5} \right) \left(\frac{h'}{h} \right)^4 - \left(\frac{1}{2} + \frac{8\pi^2}{3} \right) \left(\frac{h'}{h} \right)^2 \frac{h''}{h} + \left(\frac{\pi^2}{3} + \frac{1}{4} \right) \left(\frac{h''}{h} \right)^2 + \left(\frac{1}{16} + \frac{1}{12}\pi^2 \right) \left(\frac{h'}{h} \right)^3$$

$$\int_0^{\epsilon h} g g' dy = 0$$

$$\int_0^{\epsilon h} g g'' dy = -\left(\frac{\pi^2}{3} + \frac{1}{4} \right) \frac{h'^2}{h^2}$$

$$\int_0^{\epsilon h} g' g'' dy = \frac{h''}{h} \frac{h'}{h} \left(\frac{\pi^2}{3} + \frac{1}{4} \right) + \left(\frac{h'}{h} \right)^3 \left(-\frac{1}{4} - \frac{4}{3}\pi^2 \right)$$

$\square$

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